Gandhi Memorial College Of Education Bantalab Jammu

SPECTRUM

Ist Edition
2017

DIFFERENTIAL CALCULUS

Semester-I

Cluster University, Jammu
&
Parade College, Jammu

SHARMA PUBLICATIONS
JALANDHAR

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

32

SPECTRUM

DIFFERENTIAL CALCULUS

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

SPECTRUM
DIFFERENTIAL CALCULUS

for

Semester-I

CLUSTER UNIVERSITY, JAMMU

&

PARADE COLLEGE, JAMMU

BY

J.M. College of Education
Raipur, Bantalab
Jammu.
Acc No....3403 (D)
Dated....25/11/2024

DILEEP KUMAR RAINA
Associate Professor,
Head, Deptt. of Mathematics,
GGM Science. College,
JAMMU (J & K).

ANITA SHARMA
M.Sc., B.Ed.,
Deptt. of Mathematics,
Govt. College for Women, Gandhi Nagar,
JAMMU (J & K).

Dr. KAMALJEET KAUR
Associate Professor
Govt. College, Parade,
JAMMU (J & K).

SHOBHA GUPTA
Head, Deptt. of Mathematics,
M.A.M. College,
JAMMU (J & K).

Dr. SHALLY GUPTA
M.Sc., Ph.D.,
Deptt. of Mathematics,
Govt. Science College,
JAMMU (J & K).

SP

SHARMA PUBLICATIONS

N.D. 118, Tanda Road, Jalandhar.

Phone : 0181-2284080, 6546080, 6536080
Fax : 0181-2284080
Website : WWW.Sharmapublications.com

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

All rights Reserved
(This book or a part thereof may not be reproduced in any form without the written permission of the Publisher)

Neither Sharma Publications nor its authors guarantee the accuracy or completeness of any information published herein, and neither Sharma Publications nor its authors shall be responsible for any errors, omissions, or damages arising out of use of this information.

FIRST EDITION : 2017

PRICE : ₹ 320/-

ISBN : 978-93-5181-193-0

Every Copy of this book has the hologram of S P
Students should buy only that book which has the hologram.

Published By : Prof. D.R. Sharma for Sharma Publications, N.D. 118, Tanda Road, Jalandhar

Composed By : Nisha Handa for Accurate Centre, Jalandhar

Printed at : Alpha Printing Press, Jalandhar

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

PREFACE

It gives us immense pleasure in presenting this book for the students of Semester-I of 10 + 3 pattern.

The salient features of this book are :

(i) It has been written in simple and easy language.

(ii) Every chapter starts with an introduction and each topic has been presented in simple, lucid and systematic manner.

(iii) Solved examples have been carefully selected from the various question papers of various Indian Universities.

(iv) Wherever necessary **'Notes'** and **'Remarks'** have been given for deeper and better understanding of the subject.

We are sure that the book shall be liked both by the teachers and the taught alike. Suggestions for its improvements for all quarters shall be thankfully acknowledged and incorporated in the subsequent edition.

AUTHORS

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

CLUSTER UNIVERSITY, JAMMU

SYLLABUS

SEMESTER-I

DIFFERENTIAL Calculus

Course No : 1MATTC0101
Credits : 6

Max. Marks : 150
Internal Assessment : 30
Semester Exam : 120
(Duration : 3 Hrs.)

Unit-I

Limit and Continuity (ε and δ) definition and examples, types of discontinuity, Differentiability of functions, Successive differentiation, Leibnitz's Theorem. (24 lectures)

Unit-II

Indeterminate forms, Partial differentiation, Euler's Theorem on homogenous functions. (16 lectures)

Unit-III

Tangents & Normal, Curvature, Asymptotes, singular points (Double points), Tracing of Curves (Cartesian coordinates only). (15 lectures)

Unit-IV

Parametric representation of curves, Tracing of parametric curves, Polar coordinates and tracing of curves in polar coordinates, Area in polar coordinates. (15 lectures)

Unit-V

Definition of Maxima & Minima, Rolle's Theorem, Mean Value Theorems, Taylor's Theorem with Lagrange's and Cauchy's forms of remainder, Taylor's series and Maclaurin's Series of $\sin x$, $\cos x$, e^x, $\log(1+x)$; $(1+x)^n$.

(20 lectures)

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

PRADE COLLEGE, JAMMU

SYLLABUS

SEMESTER-I

DIFFERENTIAL Calculus

Max. Marks : 150

Unit-I

Limit and Continuity of functions on **R** (ε-δ definition) Differentiability of functions, successive differentiation and Leibnitz Theorem.

Unit-II

Partial differentiation, Euler's theorem for homogeneous functions. Maxima and Minima of Functions of two variables.

Unit-III

Envelopes, Asymptotes, Double points and Curve Tracing in Cartesian co-ordinates.

Unit-IV

Polar co-ordinates, Relation between Cartesian coordinates and Polar coordinate. Graphing Techniques in Polar forms for the curves $r = a \pm b \cos \theta$, $r = a \pm \sin \theta$, $r = \sin n\theta$, $r = \cos n\theta$, $n = 1, 2$, only.

Unit-V

Rolle's Theorem. The Mean Value Theorems. Taylors Theorem with Lagrange's and Cauchy's forms of remainder. Maclaurin's Series of $\sin x$, $\cos x$, $\tan x$, $\sec x$, e^x, $\log(1+x)$; $(1+x)^m$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

CONTENTS

1.	Limit and Continuity of a Function (Using ε - δ def.)	1 – 40
2.	Differentiability of Functions	41 – 53
3.	Successive Differentiation	54 – 94
4.	Indeterminate Forms	95 – 122
5.	Partial Differentiation	123 – 156
6.	Tangents and Normal	157 – 186
7.	Curvature	187 – 216
8.	Asymptotes	217 – 239
9.	Double Points	240 – 255
10.	Tracing of Curves in Cartesian Coordinates	256 – 281
11.	Tracing of Parametric Curves	282 – 286
12.	Polar Coordinates	287 – 299
13.	Graphing Techniques in Polar Coordinates	300 – 309
14.	Maxima and Minima	310 – 341
15.	General Theorems	342 – 370
16.	Envelops	371 – 392

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

1

LIMIT AND CONTINUITY OF A FUNCTION (*Using* ε-δ *def.*)

Art-1. Introduction

Limits play a very important role in calculus. In fact, limits and continuity provide the foundation for theory of calculus.

We have already discussed various types of functions and sketched their graphs. The graphs of some functions are continuous *i.e.*, without any break whereas, some of them have shown discontinuous nature or have breaks at some of the points, *e.g.*, $\sin x$, $\cos x$, $|x|$ have continuous graphs where $[x]$, $\tan x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$ etc. are discontinuous functions for all x. The behaviour of a function, that how smoothly graph of a function varies near about a point of its domain is of great importance. Limit of a function helps us this course of study.

Art-2. Neighbourhood of a Point

Let c be any real number and $a, b \in \mathbf{R}$ such that $a < c < b$ then the open interval (a, b) is called a neighbourhood of c. In other words, any open interval (a, b) containing $c \in \mathbf{R}$ is called neighbourhood of c.

The size (or length) of a neighbourhood (a, b) is measured by $|b - a|$. The neighbourhood (a, b) is called symmetric neighbourhood of c iff $|c - a| = |c - b|$. Thus for a given $\delta > 0$, $(c-\delta, c+\delta) = \{x \in \mathbf{R} : |x - c| < \delta\}$ is called δ-neighbourhood of c and is symmetrical about c.

Deleted neighbourhood

Any neighbourhood of c not containing the point c is called a deleted neighbourhood of c. Thus if $c \in (a, b)$, then $(a, c) \cup (c, b)$ is a deleted neighbourhood of c. The set

$$\{x \in \mathbf{R} : 0 < |x - c| < \delta\} = \{x \in \mathbf{R} : |x - c| < \delta \text{ and } x \neq c\}$$

$$= (c - \delta, c) \cup (c, c + \delta) \text{ is a deleted neighbourhood of } c.$$

One sided neighbourhood

Let c be any real number and $a, b \in \mathbf{R}$ such that $a < c < b$ then $(a, c]$ is called a **left neighbourhood** and $[c, b)$ is called a **right neighbourhood** of c. Similarly $(c - \delta, c]$ and $[c, c + \delta)$ are called left and right δ-neighbourhood of c. Also $(c-\delta, c)$ and $(c, c + \delta)$ are called left and right deleted δ-neighbourhood of the point c.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Art-3. Meaning of $x \to a$ (x approaches a)

$x \to a$ means $x \neq a$ but $|x - a|$ is very small. So $x \to a$ means there exists a positive number $\delta > 0$, however small, such that $0 < |x - a| < \delta$

or $x \in (a - \delta, a + \delta)$ and $x \neq a$

or $x \in (a - \delta, a) \cup (a, a + \delta)$

if $x \in (a - \delta, a)$ only, then we say that x tends to a from the left and we write it as $x \to a-$ or $x \to a - 0$. Some authors also write like $x \to a^-$. Similarly when $x \in (a, a + \delta)$ only, then we say that x tends to a from right. We write it as $x \to a+$ or $x \to a^+$ or $x \to a + 0$.

Art-4. Limit of a Function

Having studied the concept of a neighbourhood of a point c, we are in a position to understand the most important fact of calculus, so called limit. In a vague way, a function f is said to have a limit l as x is approaching a point c if $f(x)$ is in neighbourhood of l for x lying in the deleted neighbourhood of c. Let us study one illustration to make the above argument clear.

Consider the function $f(x) = \dfrac{x^2 - 1}{x - 1}$, for $x \neq 1$

It is clear that $f(1)$ is undefined. However, when x approaches 1 and is never equal to 1, the values of $f(x)$ are particularly near 2 as will be clear from the following table :

Table

x	0·9	·99	·999	·9999	1·0001	1·001	1·01	1·1
$f(x)$	1·9	1·99	1·999	1·9999	2·0001	2·001	2·01	2·1

The same fact can be expressed by saying that the difference between $f(x)$ and 2 is as small as possible for x sufficiently near 1 and never equal to 1. More precisely, if $\varepsilon > 0$, however small, $\exists$ a + ve real number δ (very-very small) s.t.

$$|f(x) - 2| < \varepsilon \quad \text{for} \quad 0 < |x - 1| < \delta$$

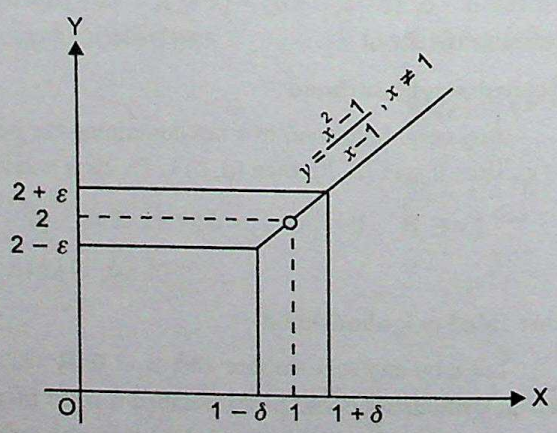
We have said that x is near 1. To make it more clear, if x is on the left of 1 and is near 1, we say $f(x)$ has left limit 2

$\therefore$ for left limit we have

$|f(x) - 2| < \varepsilon$ for $1 - \delta < x < 1$ where $\varepsilon, \delta > 0$ and very small

Similarly if x is on the right of 1 and is near 1, we say $f(x)$ has right limit, 2.

$\therefore$ for right limit we have



$$|f(x) - 2| < \varepsilon \quad \text{for} \quad 1 < x < 1 + \delta \text{ where } \varepsilon, \delta > 0 \text{ and very small.}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

When both the sides are considered simultaneously we say $f(x)$ has the limit 2 as x approaches 1.

$\therefore$ for limit we have $|f(x)-2|<\varepsilon$ for $0<|x-1|<\delta$

where $\varepsilon,\ \delta>0$ and very small.

Geometrically, we represent the above facts like this

Left limit : A function f is said to have a left limit l as $x \to a-$, written as $\underset{x\to a-}{\text{Lt}} f(x)=l$, if given $\varepsilon>0$, however small, there exists a positive real number $\delta(\varepsilon)$ such that $|f(x)-l|<\varepsilon$ for $a-\delta<x<a$

Note. $\delta(\varepsilon)$ means δ depends upon ε.

Right limit. A function f is said to have a right limit l as $x \to a+$, written as $\underset{x\to a+}{\text{Lt}} f(x)=l$, if given $\varepsilon>0$, however small, there exists a positive real number $\delta(\varepsilon)$ such that $|f(x)-l|<\varepsilon$ for $a<x<a+\delta$.

Limit. A function f is said to have a limit l as $x \to a$, written as, $\underset{x\to a}{\text{Lt}} f(x)=l$, if given $\varepsilon>0$, however small, there exists a positive real number $\delta(\varepsilon)$ such that

$$|f(x)-l|<\varepsilon \quad \text{for} \quad 0<|x-a|<\delta.$$

Note 1. For a function f to have a limit as $x \to a$, it is necessary that f is defined at $x=a$.

Note 2. Left limit (or left hand limit) of f as $x \to a-$ is also called a **limit from below** and is denoted by $f(a-)$ or $f(a-0)$. Similarly right hand limit of f as $x \to a+$ is also called a **limit from above** and is denoted by $f(a+)$ or $f(a+0)$.

Note 3. We will be writing

$$\underset{x\to a-}{\text{Lt}} f(x)=\underset{h\to 0+}{\text{Lt}} f(a-h)$$

and $$\underset{x\to a+}{\text{Lt}} f(x)=\underset{h\to 0+}{\text{Lt}} f(a+h)$$

Note 4. If $\underset{x\to a-}{\text{Lt}} f(x)\neq \underset{x\to a+}{\text{Lt}} f(x)$ or even one of them does not exist, then $\underset{x\to a}{\text{Lt}} f(x)$ does not exist.

Note 5. If $\underset{x\to a-}{\text{Lt}} f(x)=\underset{x\to a+}{\text{Lt}} f(x)=l$, then we say that $\underset{x\to a}{\text{Lt}} f(x)$ exists and is equal to l. We write it as $\underset{x\to a}{\text{Lt}} f(x)=l$.

Art-5. Prove that $\underset{x\to a}{\text{Lt}} f(x)$ exists iff $\underset{x\to a-}{\text{Lt}} f(x)$, $\underset{x\to a+}{\text{Lt}} f(x)$ both exists and are equal.

Proof. (*i*) Assume that $\underset{x\to a}{\text{Lt}} f(x)$ exists and is equal to l.

$$\therefore \quad \underset{x\to a}{\text{Lt}} f(x)=l$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$\Rightarrow$ given $\varepsilon > 0$, however small, there exists a positive real number $\delta(\varepsilon)$ such that

$$|f(x) - l| < \varepsilon \quad \text{for} \quad 0 < |x - a| < \delta$$

$\Rightarrow$ $|f(x) - l| < \varepsilon$ for $a - \delta < x < a$

and $|f(x) - l| < \varepsilon$ for $a < x < a + \delta$

$\Rightarrow$ $\underset{x \to a-}{\text{Lt}} f(x)$ and $\underset{x \to a+}{\text{Lt}} f(x)$ both exist and are equal

(*ii*) Assume that $\underset{x \to a-}{\text{Lt}} f(x)$ and $\underset{x \to a+}{\text{Lt}} f(x)$ both exists and are equal, say l.

$\therefore$ $\underset{x \to a-}{\text{Lt}} f(x) = l$ and $\underset{x \to a+}{\text{Lt}} f(x) = l$

Now $\underset{x \to a-}{\text{Lt}} f(x) = l$

$\Rightarrow$ given $\varepsilon > 0$, however small, there exists a positive real number $\delta_1(\varepsilon)$ such that

$$|f(x) - l| < \varepsilon \quad \text{for} \quad a - \delta_1 < x < a \qquad \ldots(1)$$

Again $\underset{x \to a+}{\text{Lt}} f(x) = l$

$\Rightarrow$ given $\varepsilon > 0$, however small, there exists a positive real number $\delta_2(\varepsilon)$ such that

$$|f(x) - l| < \varepsilon \quad \text{for} \quad a < x < a + \delta_2 \qquad \ldots(2)$$

Let $\delta = \min. \{\delta_1, \delta_2\}$

$\therefore$ from (1) and (2), we get,

$$|f(x) - l| < \varepsilon \quad \text{for} \quad a - \delta < x < a$$

and $|f(x) - l| < \varepsilon$ for $a < x < a + \delta$

$\Rightarrow$ $|f(x) - l| < \varepsilon$ for $0 < |x - a| < \delta$

$\Rightarrow$ $\underset{x \to a}{\text{Lt}} f(x) = l$

Art-6. If $\underset{x \to a}{\text{Lt}} f(x)$ exists finitely, then prove that it is unique.

Proof. If possible, suppose

$$\underset{x \to a}{\text{Lt}} f(x) = l \text{ and } \underset{x \to a}{\text{Lt}} f(x) = l' \text{ where } l \neq l'$$

Without any loss of generality, we assume that $l' > l$.

Let $$\varepsilon = \frac{l' - l}{2} > 0.$$

Now $\underset{x \to a}{\text{Lt}} f(x) = l$ $\Rightarrow$ given $\varepsilon > 0$, however small,

we can find a positive real number $\delta_1(\varepsilon)$ such that

$$|f(x) - l| < \varepsilon \quad \text{for} \quad 0 < |x - a| < \delta_1 \qquad \ldots(1)$$

Again $\underset{x \to a}{\text{Lt}} f(x) = l'$ $\Rightarrow$ given $\varepsilon > 0$, however small, we can find a positive real number $\delta_2(\varepsilon)$ such that

$$|f(x) - l'| < \varepsilon \text{ for } 0 < |x - a| < \delta_2 \qquad \ldots(2)$$

Let $\delta = \min\{\delta_1, \delta_2\}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$\therefore$ from (1) and (2), we get,

$$|f(x)-l|<\varepsilon \quad \text{for} \quad 0<|x-a|<\delta \qquad \ldots(3)$$

and $|f(x)-l'|<\varepsilon$ for $0<|x-a|<\delta$ $\qquad \ldots(4)$

Now $l'-l=|l'-l|$ $\qquad [\because l'-l>0]$

$$=|l'-f(x)+f(x)-l| = |\{l'-f(x)\}+\{f(x)-l\}|$$

$$\leq |l'-f(x)|+|f(x)-l| \qquad [\because |a+b|\leq|a|+|b|]$$

$$=|f(x)-l'|+|f(x)-l| \qquad [\because |-x|=|x|]$$

$$<\varepsilon+\varepsilon \qquad [\because \text{of (3) and (4)}]$$

$$=2\varepsilon=l'-l \qquad \left[\because \varepsilon=\frac{l'-l}{2}\right]$$

$\therefore$ $l'-l<l'-l$, which is a contradiction.

$\therefore$ our supposition is wrong.

$\therefore$ if $\underset{x\to a}{\text{Lt}} f(x)$ exists finitely, then it is unique.

Another Form : If $\underset{x\to a}{\text{Lt}} f(x)=l$ are $\underset{x\to a}{\text{Lt}} f(x)=l'$, then prove that $l=l'$.

Note. Student can prove it easily that

(*i*) If $\underset{x\to a-}{\text{Lt}} f(x)$ exists finitely, then it is unique.

(*ii*) If $\underset{x\to a+}{\text{Lt}} f(x)$ exists finitely, then it is unique.

Art-7. If $\underset{x\to a}{\text{Lt}} f(x)=l$, then f is bounded in deleted neighbourhood of a.

Proof. Let k be any positive real number greater than $|l|$.

Let $\varepsilon=k-|l|>0$

Now $\underset{x\to a}{\text{Lt}} f(x)=l \Rightarrow$ given $\varepsilon>0$, however small, we can find a positive real number δ (ε) such that

$$|f(x)-l|<\varepsilon \quad \text{for} \quad 0<|x-a|<\delta \qquad \ldots(1)$$

Now $|f(x)|=|\{f(x)-l\}+l| \leq |f(x)-l|+|l|$

$$<\varepsilon+|l| \qquad [\because \text{of (1)}]$$

$$=k-|l|+|l| \qquad [\because \varepsilon=k-|l|]$$

$$=k$$

$\therefore$ $|f(x)|<k$ for $0<|x-a|<\delta$

$\therefore$ f is bounded in deleted neighbourhood of a.

Note 1. Converse of this theorem is not true i.e. bounded function in deleted nbd. of c may not have limit at c.

Note 2. If $f(x)$ is not bounded in a deleted nbd. of c then $\underset{x\to c}{\text{Lt}} f(x)$ does not exist.

Note 3. The above theorem is also true for one sided limits.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Art-8. Cauchy's Criterion

The $\underset{x \to a}{\text{Lt}} f(x)$ exists finitely if given $\varepsilon > 0$, however small, there exists a + ve real number δ, such that $|f(x_1) - f(x_2)| < \varepsilon$ for $0 < |x_i - a| < \delta$ where $i = 1, 2$.

Proof. Let $\underset{x \to a}{\text{Lt}} f(x) = l$ (exists finitely)

$\therefore$ given $\varepsilon > 0$, however small, these exists a + ve real number δ such that

$$|f(x_1) - l| < \frac{\varepsilon}{2} \quad \text{for} \quad 0 < |x_1 - \alpha| < \delta \qquad \ldots(1)$$

$$\text{and} \quad |f(x_2) - l| < \frac{\varepsilon}{2} \quad \text{for} \quad 0 < |x_2 - \alpha| < \delta \qquad \ldots(2)$$

Now $|f(x_1) - f(x_2)| = |f(x_1) - l + l - f(x_2)| = |\{f(x_1) - l\} - \{f(x_2) - l\}|$

$$\leq |f(x_1) - l| + |f(x_2) - l|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for } \begin{matrix} 0 < |x_1 - \alpha| < \delta \\ 0 < |x_2 - \alpha| < \delta \end{matrix} \qquad [\because \text{ of (1) and (2)}]$$

Hence the result.

Note : The above result is very useful. If in a problem, we are able to show that

$$|f(x_1) - f(x_2)| > \varepsilon \text{ for } 0 < |x_i - \alpha| < \delta \text{ where } i = 1, 2$$

Then we can say $\underset{x \to a}{\text{Lt}} f(x)$ does not exist

ILLUSTRATIVE EXAMPLES

Example 1. By use of definition of limit, show that $\underset{x \to 2}{\text{Lt}} (4x - 5) = 3$.

Sol. Let $f(x) = 4x - 5$, $l = 3$

$\therefore$ $|f(x) - l| = |(4x - 5) - 3| = |4x - 8| = 4|x - 2|$

Let $\varepsilon > 0$, however small, be given.

Now $|f(x) - l| < \varepsilon$ whenever $4|x - 2| < \varepsilon$

i.e., $|x - 2| < \dfrac{\varepsilon}{4} = \delta$ (say)

$\therefore$ given $\varepsilon > 0$, however small, we can find a positive number $\delta(\varepsilon)$ such that

$|f(x) - l| < \varepsilon$ for $0 < |x - 2| < \delta$

$\therefore$ $\underset{x \to 2}{\text{Lt}} f(x) = l$, or $\underset{x \to 2}{\text{Lt}} (4x - 5) = 3$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 2. Using definition of limit, prove that $\underset{x\to 3}{\text{Lt}} \dfrac{x^2-9}{x-3} = 6.$

Sol. Let $f(x) = \dfrac{x^2-9}{x-3}$, $l = 6$

$$\therefore \ |f(x) - l| = \left|\frac{x^2-9}{x-3} - 6\right| = |x+3-6| = |x-3|$$

Let $\varepsilon > 0$, however small, be given.

Now $|f(x) - l| < \varepsilon$ whenever $|x-3| < \varepsilon = \delta$ (say)

$\therefore$ given $\varepsilon > 0$, however small, we can find a positive number $\delta\ (\varepsilon)$ such that

$|f(x) - l| < \varepsilon$ for $0 < |x-3| < \delta$

$$\therefore \ \underset{x\to 3}{\text{Lt}} f(x) = l \quad \text{or} \quad \underset{x\to 2}{\text{Lt}} \frac{x^2-9}{x-3} = 6.$$

Example 3. Let $f(x) = \dfrac{1-x}{1+x}$, $0 < x < 1$. Find a real number $\delta > 0$ such that

$$\left|f(x) - \frac{1}{3}\right| < \frac{1}{10} \text{ for } \left|x - \frac{1}{2}\right| < \delta.$$

Sol. Here $f(x) = \dfrac{1-x}{1+x}$, $0 < x < 1$

$$\left|f(x) - \frac{1}{3}\right| = \left|\frac{1-x}{1+x} - \frac{1}{3}\right| = \left|\frac{3-3x-1-x}{3(1+x)}\right| = \left|\frac{-4x+2}{3(1+x)}\right| = \frac{4}{3(1+x)}\left|x - \frac{1}{2}\right|$$

[$\because$ $0 < x < 1 \Rightarrow 1 + x > 0$ and hence $|1+x| = 1+x$

Also $\left|-x + \dfrac{1}{2}\right| = \left|x - \dfrac{1}{2}\right|$]

We want to find δ such that $\left|f(x) - \dfrac{1}{3}\right| < \dfrac{1}{10}$ for $\left|x - \dfrac{1}{2}\right| < \delta$

i.e., $\dfrac{4}{3(1+x)}\left|x - \dfrac{1}{2}\right| < \dfrac{1}{10}$ for $\left|x - \dfrac{1}{2}\right| < \delta$

i.e., $\left|x - \dfrac{1}{2}\right| < \dfrac{3}{40}(1+x)$ for $0 < \left|x - \dfrac{1}{2}\right| < \delta$

If we take $\delta = \dfrac{3}{40}$, we have

$$\left|x - \frac{1}{2}\right| < \delta \ \ i.e., \left|x - \frac{1}{2}\right| < \frac{3}{40}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\Rightarrow \quad \left| x - \frac{1}{2} \right| < \frac{3}{40}(1+x) \qquad [\,1 < 1 + x < 2 \text{ as } 0 < x < 1]$$

$$\Rightarrow \quad \frac{4}{3(1+x)} \left| x - \frac{1}{2} \right| < \frac{1}{10}$$

Hence $\left| f(x) - \frac{1}{3} \right| < \frac{1}{10}$ for $\left| x - \frac{1}{2} \right| < \delta$

$\therefore$ required number $= \frac{3}{40}$.

Example 4. (*a*) Let $f(x) = \begin{cases} 2 + x, & x \text{ rational} \\ 2{\cdot}5 - x, & x \text{ irrational} \end{cases}$

then prove that $\underset{x \to \cdot 25}{\text{Lt}} f(x) = 2{\cdot}25$

(*b*) Let $f(x) = \begin{cases} 5 + x, & x \text{ rational} \\ 5.5 - x, & x \text{ irrational} \end{cases}$

then prove that $\underset{x \to \cdot 25}{\text{Lt}} f(x) = 5{\cdot}25$.

Sol. (*a*) Here $f(x) = \begin{cases} 2 + x, & x \text{ rational} \\ 2{\cdot}5 - x, & x \text{ irrational} \end{cases}$

Also $l = 2{\cdot}25$

$\therefore$ when x is rational.

$$| f(x) - l | = | 2 + x - 2{\cdot}25 | = | x - 0{\cdot}25 |$$

Again when x is irrational

$$| f(x) - l | = | 2{\cdot}5 - x - 2{\cdot}25 | = | 0{\cdot}25 - x | = | x - 0{\cdot}25 |$$

$\therefore \quad | f(x) - l | = | x - 0{\cdot}25 | \quad \forall x \in \mathbf{R}$

Let $\varepsilon > 0$, however small, be given

Now $| f(x) - l | < \varepsilon$ whenever $| x - \cdot 25 | < \varepsilon = \delta$ (say)

$\therefore$ given $\varepsilon > 0$, however small, we can find a positive number δ (ε) such that

$$| f(x) - l | < \varepsilon \text{ for } 0 < | x - \cdot 25 | < \delta$$

$\therefore \quad \underset{x \to \cdot 25}{\text{Lt}} f(x) = l$, or $\underset{x \to \cdot 25}{\text{Lt}} f(x) = 2{\cdot}25$

(*b*) Here $f(x) = \begin{cases} 5 + x, & x \text{ rational} \\ 5.5 - x, & x \text{ irrational} \end{cases}$

Also $l = 5{\cdot}25$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$\therefore$ when x is rational.

$$|f(x)-l| = |5+x-5{\cdot}25| = |x-0{\cdot}25|$$

Again when x is irrational

$$|f(x)-l| = |5{\cdot}5-x-5{\cdot}25| = |0{\cdot}25-x| = |x-0{\cdot}25|$$

$\therefore$ $|f(x)-l| = |x-0{\cdot}25| \quad \forall x \in \mathbf{R}$

Let $\varepsilon > 0$, however small, be given

Now $|f(x)-l| < \varepsilon$ whenever $|x-{\cdot}25| < \varepsilon = \delta$ (say)

$\therefore$ given $\varepsilon > 0$, however small, we can find a positive number $\delta(\varepsilon)$ such that

$$|f(x)-l| < \varepsilon \text{ for } 0 < |x-{\cdot}25| < \delta$$

$\therefore$ $\underset{x \to {\cdot}25}{\text{Lt}} f(x) = l$, or $\underset{x \to {\cdot}25}{\text{Lt}} f(x) = 5{\cdot}25$

Example 5. (*a*) Let $f(x) = \dfrac{|x|}{x}$ for all x except 0. Prove that $\underset{x \to 0}{\text{Lt}} f(x)$ does not exist.

(*b*) Let $f(x) = \begin{cases} 1 \text{ if } x \in \mathbf{Q} \\ -1 \text{ if } x \in \mathbf{R}-\mathbf{Q} \end{cases}$

Prove, using definition, that $\underset{x \to c}{\text{Lt}} f(x)$ does not exist.

Sol. (*a*) Let $\varepsilon > 0$, however small.

We choose $0 < \varepsilon < 1$

Let $\delta > 0$, a + ve real

$\because$ $x \to 0$

$\therefore$ we take $0 < |x_1 - 0| < \delta$

$0 < |x_2 - 0| < \delta$ where $x_1 > 0$ and $x_2 < 0$

$$\therefore \ |f(x_1)-f(x_2)| = \left|\frac{|x_1|}{x_1} - \frac{|x_2|}{x_2}\right| = \left|\frac{x_1}{x_1} - \frac{-x_2}{x_2}\right| = |1+1| = 2 > 1 > \varepsilon$$

$\therefore$ $|f(x_1)-f(x_2)| > \varepsilon$ where $0 < |x_1 - 0| < \delta$

$0 < |x_2 - 0| < \delta$

(and $0 < \varepsilon < 1$)

$\therefore$ $\underset{x \to 0}{\text{Lt}} f(x)$ does not exist.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Foundation Bentalab, Amritsar

(*b*) Let $\varepsilon > 0$, however small.

We choose $0 < \varepsilon < 1$

Let $\delta > 0$, a + ve real

We take $0 < |x_1 - c| < \delta$

$0 < |x_2 - c| < \delta$

and $x_1 \in \mathbf{Q},\ x_2 \in \mathbf{R} - \mathbf{Q}$

Now $|f(x_1) - f(x_2)| = |1 - (-1)| = 2 > 1 > \varepsilon$

$\therefore\ |f(x_1) - f(x_2)| > \varepsilon$ for $0 < |x_1 - c| < \delta$

$0 < |x_2 - c| < \delta$

$\therefore\ \underset{x \to c}{\text{Lt}}\ f(x)$ does not exist.

Remark : Let $f(x) = \begin{cases} 1 \text{ if } x \in \mathbf{Q} \\ -1 \text{ if } x \in \mathbf{R} - \mathbf{Q} \end{cases}$; $g(x) = \begin{cases} -1 \text{ if } x \in \mathbf{Q} \\ 1 \text{ if } x \in \mathbf{R} - \mathbf{Q} \end{cases}$

Then $\underset{x \to c}{\text{Lt}}\ f(x)$ and $\underset{x \to c}{\text{Lt}}\ g(x)$ do not exist

Now $(f + g)(x) = 0 \quad \forall\ x$

$\therefore\ \underset{x \to c}{\text{Lt}}\ (f + g)(x) = 0$ (exists)

$(f.g)(x) = -1 \quad \forall\ x$

$\underset{x \to c}{\text{Lt}}\ (f.g)(x) = -1$ (exists)

Note : If $\underset{x \to c}{\text{Lt}}\ f(x) = l$, then $\underset{x \to c}{\text{Lt}}\ |f(x)| = |l|$. Is the converse true ? Give reasons.

Sol. Since $\underset{x \to c}{\text{Lt}}\ f(x) = l$

$\therefore$ given $\varepsilon > 0$, however small, we can find a positive real number $\delta(\varepsilon)$ such that

$|f(x) - l| < \varepsilon$ for $0 < |x - c| < \delta$ ...(1)

Now $\big||f(x)| - |l|\big| \le |f(x) - l|$ ...(2)

$[\because \big||a| - |b|\big| \le |a - b|]$

From (1) and (2), we get

$\big||f(x)| - |l|\big| < \varepsilon$ for $0 < |x - c| < \delta$

$\therefore\ \underset{x \to c}{\text{Lt}}\ |f(x)| = |l|$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Its converse is not true.

For example,

Consider $f(x) = \begin{cases} 1, & x \text{ is rational} \\ -1, & x \text{ is irrational} \end{cases}$

$\therefore \quad |f(x)| = 1 \ \forall \ x \in \mathbf{R}$

$\therefore \quad \underset{x \to c}{\text{Lt}} \ |f(x)| = \underset{x \to c}{\text{Lt}} \ 1 = 1$

$\therefore \quad \underset{x \to c}{\text{Lt}} \ |f(x)|$ exists at all $c \in \mathbf{R}$ but $\underset{x \to c}{\text{Lt}} \ f(x)$ does not exist at any $c \in \mathbf{R}$.

Example 6. Prove that $\underset{x \to a}{\text{Lt}} \dfrac{1}{x-a}$ does not exist.

Sol. Let $f(x) = \dfrac{1}{x-a}$

Let $\varepsilon > 0$, however small.

We choose $0 < \varepsilon < 1$

Let $\delta > 0$, a positive real.

Choose a positive integer n such that $\dfrac{1}{n} < \delta$ *i.e.*, $n > \dfrac{1}{\delta}$

$\therefore \quad \dfrac{1}{2n} < \dfrac{1}{n} < \delta$

Let $x_1 = a + \dfrac{1}{n}$, $x_2 = a + \dfrac{1}{2n}$ be two real numbers satisfying $0 < |x - a| < \delta$.

Then $f(x_1) = n$, $f(x_2) = 2n$

$\therefore \quad |f(x_1) - f(x_2)| = |n - 2n| = |-n| = n > 1 > \varepsilon$

where x_1 and x_2 are two real numbers satisfying $0 < |x - a| < \delta$

$\therefore \quad \underset{x \to a}{\text{Lt}} \ f(x) = \underset{x \to a}{\text{Lt}} \dfrac{1}{x-a}$ does not exist.

Cor. Taking $a = 1, 2$, we get

(*i*) $\underset{x \to 1}{\text{Lt}} \left(\dfrac{1}{x-1}\right)$ does not exist.

(*ii*) $\underset{x \to 2}{\text{Lt}} \left(\dfrac{1}{x-2}\right)$ does not exist.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 7. Prove that $\underset{x \to c-}{\text{Lt}} \dfrac{1}{x-c}$ does not exist.

Sol. Let $f(x) = \dfrac{1}{x-c}$

Let $\varepsilon > 0$, however small.

We choose $0 < \varepsilon < 1$

Let $\delta > 0$, a positive real.

Choose a positive integer n such that $\dfrac{1}{n} < \delta$ *i.e.*, $n > \dfrac{1}{\delta}$

$\therefore \quad \dfrac{1}{2n} < \dfrac{1}{n} < \delta$

Let $x_1 = c - \dfrac{1}{n}, x_2 = c - \dfrac{1}{2n}$ be two real numbers satisfying $x_1, x_2 \in (c-\delta, c)$.

Then $f(x_1) = n,\ f(x_2) = 2n$

$\therefore \quad |f(x_1) - f(x_2)| = |n - 2n| = |-n| = n > 1 > \varepsilon$

where x_1 and x_2 are two real numbers satisfying $x_1, x_2 \in (c-\delta, c)$

$\therefore \quad \underset{x \to c-}{\text{Lt}} f(x) = \underset{x \to c-}{\text{Lt}} \dfrac{1}{x-c}$ does not exist.

Example 8. Show that $\underset{x \to 0}{\text{Lt}} \sin \dfrac{1}{x}$ does not exist.

Sol. Let $f(x) = \sin \dfrac{1}{x}$

Let $0 < \varepsilon < 1$ and δ be a positive real number.

Let n be a positive integer such that

$x_1 = \dfrac{1}{2n\pi},\ x_2 = \dfrac{1}{2n\pi + \dfrac{\pi}{2}}$ be two real numbers satisfying $0 < |x| < \delta$. Then

$$f(x_1) = \sin 2n\pi = 0,\ f(x_2) = \sin\left(2n\pi + \frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$$

$\therefore \quad |f(x_1) - f(x_2)| = |0 - 1| = 1 > \varepsilon$

where x_1 and x_2 are two real numbers satisfying $0 < |x| < \delta$.

$\therefore \quad \underset{x \to 0}{\text{Lt}} f(x) = \underset{x \to 0}{\text{Lt}} \sin \dfrac{1}{x}$ does not exist.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 9. Prove that $\underset{x\to -\frac{5}{2}}{\text{Lt}} \dfrac{1}{2x+5}$ does not exist.

Sol. If possible, suppose that $\underset{x\to -\frac{5}{2}}{\text{Lt}} \dfrac{1}{2x+5} = l$

$\therefore$ given $\varepsilon > 0$, however small, there exists a positive real number $\delta(\varepsilon)$ such that

$$\left|\frac{1}{2x+5} - l\right| < \varepsilon \text{ for all } x \text{ in } 0 < \left|x - \left(-\frac{5}{2}\right)\right| < \delta$$

$$\Rightarrow \left|\frac{1}{2x+5} - l\right| < \varepsilon \text{ for } 0 < \left|x + \frac{5}{2}\right| < \delta \qquad \ldots(1)$$

Choose a positive integer n such that $n > \dfrac{1}{\delta}$ *i.e.*, $\dfrac{1}{n} < \delta$. Then $\dfrac{1}{2n} < \dfrac{1}{n} < \delta$ so that $-\dfrac{5}{2} + \dfrac{1}{n}$ and $-\dfrac{5}{2} + \dfrac{1}{2n}$ are real numbers in $0 < \left|x + \dfrac{5}{2}\right| < \delta$

$\therefore$ (1) must hold for $x = -\dfrac{5}{2} + \dfrac{1}{n}$ and $x = -\dfrac{5}{2} + \dfrac{1}{2n}$

$$\therefore \left|\frac{1}{2\left(-\frac{5}{2}+\frac{1}{n}\right)+5} - l\right| < \varepsilon \text{ and } \left|\frac{1}{2\left(-\frac{5}{2}+\frac{1}{2n}\right)+5} - l\right| < \varepsilon$$

$$\Rightarrow \left|\frac{n}{2} - l\right| < \varepsilon \text{ and } |n - l| < \varepsilon$$

Now $\left|\dfrac{n}{2} - n\right| = \left|\left(\dfrac{n}{2} - l\right) - (n - l)\right| \le \left|\dfrac{n}{2} - l\right| + |n - l| < \varepsilon + \varepsilon = 2\varepsilon$

$\therefore \left|-\dfrac{n}{2}\right| < 2\varepsilon \Rightarrow \dfrac{n}{2} < 2\varepsilon \Rightarrow n < 4\varepsilon$, which is not so if we take $\varepsilon < \dfrac{1}{4}$

$\therefore$ our supposition is wrong.

$\therefore \underset{x\to -\frac{3}{2}}{\text{Lt}} \dfrac{1}{2x+5}$ does not exist.

Example 10. Using definition of limit, show that

(*i*) $\underset{x\to c}{\text{Lt}} (x-c)\sin\dfrac{1}{x-c} = 0$ (*ii*) $\underset{x\to c}{\text{Lt}} (x-c)\cos\dfrac{1}{x-c} = 0.$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Sol. (*i*) Let $f(x) = (x - c) \sin \dfrac{1}{x-c}$.

Let $\varepsilon > 0$, however small, be given

Now $|f(x) - 0| = \left|(x-c)\sin\dfrac{1}{x-c} - 0\right| = \left|(x-c)\sin\dfrac{1}{x-c}\right|$

$\leq |x - c|$ $\qquad \left[\because \left|\sin\dfrac{1}{x-c}\right| \leq 1 \text{ for } x \neq c\right]$

$< \varepsilon$ whenever $0 < |x - c| < \varepsilon$

$\therefore \quad |f(x) - 0| < \varepsilon$ whenever $0 < |x - c| < \delta$, where $\delta = \varepsilon$

$\Rightarrow \quad \underset{x \to c}{\text{Lt}} f(x) = 0$

$\therefore \quad \underset{x \to c}{\text{Lt}} (x-c)\sin\dfrac{1}{x-c} = 0.$

(*ii*) Let $f(x) = (x - c) \cos \dfrac{1}{x-c}$.

Let $\varepsilon > 0$, however small, be given

Now $|f(x) - 0| = \left|(x-c)\cos\dfrac{1}{x-c} - 0\right| = \left|(x-c)\cos\dfrac{1}{x-c}\right|$

$\leq |x - c|$ $\qquad \left[\because \left|\cos\dfrac{1}{x-c}\right| \leq 1 \text{ for } x \neq c\right]$

$< \varepsilon$ whenever $0 < |x - c| < \varepsilon$

$\therefore \quad |f(x) - 0| < \varepsilon$ whenever $0 < |x - c| < \delta$, where $\delta = \varepsilon$

$\Rightarrow \quad \underset{x \to c}{\text{Lt}} f(x) = 0$

$\therefore \quad \underset{x \to c}{\text{Lt}} (x-c)\cos\dfrac{1}{x-c} = 0.$

EXERCISE 1 (*a*)

1. By use of definition of limit, show that $\underset{x \to -1}{\text{Lt}} (2x + 3) = 1$.

2. Use definition of limit to prove that

(*i*) $\underset{x \to 3}{\text{Lt}} (11 - 2x) = 5$ (*ii*) $\underset{x \to 3}{\text{Lt}} (1 - 3x) = -8$

(*iii*) $\underset{x \to 2}{\text{Lt}} (2 - 3x) = -4$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

3. Using definition of limit, prove that

(*i*) $\underset{x \to 1}{\text{Lt}} \dfrac{x^2 - 1}{x - 1} = 2.$

(*ii*) $\underset{x \to 2}{\text{Lt}} \dfrac{x^2 - 4}{x - 2} = 4.$

4. If $f(x) = \begin{cases} x, & x \in \mathbf{Q} \\ -x, & x \in \mathbf{R} - \mathbf{Q} \end{cases}$

then prove that $\underset{x \to 0}{\text{Lt}} f(x) = 0.$

5. Let $f(x) = \begin{cases} 0 & \text{if} \quad x \text{ is irrational} \\ 1 & \text{if} \quad x \text{ is rational} \end{cases}$

show that $\underset{x \to c}{\text{Lt}} f(x)$ does not exist for any $c \in \mathbf{R}$.

6. If $f(x) = \begin{cases} 2, & x \text{ is rational} \\ 2{\cdot}5, & x \text{ is irrational} \end{cases}$

then prove that $\underset{x \to a-}{\text{Lt}} f(x)$ and $\underset{x \to a+}{\text{Lt}} f(x)$ do not exist where a is any real number.

7. Let $f(x) = \begin{cases} 0, & x \geq 0 \\ 1, & x < 0 \end{cases}$ and $g(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$

then prove, by definition, that

(*i*) $\underset{x \to 0}{\text{Lt}} f(x)$ does not exist

(*ii*) $\underset{x \to 0}{\text{Lt}} g(x)$ does not exist

(*iii*) $\underset{x \to 0}{\text{Lt}} (f + g)(x) = 1$

(*iv*) $\underset{x \to 0}{\text{Lt}} (f \cdot g)(x) = 0$

8. Prove that $\underset{x \to c+}{\text{Lt}} \dfrac{1}{x - c}$ does not exist.

9. Prove that the following limits do not exist.

(*i*) $\underset{x \to 0}{\text{Lt}} \dfrac{1}{x}$

(*ii*) $\underset{x \to 0}{\text{Lt}} \dfrac{1}{x^2}$

(*iii*) $\underset{x \to -\frac{3}{2}}{\text{Lt}} \dfrac{1}{2x + 3}$

(*iv*) $\underset{x \to 0}{\text{Lt}} \cos \dfrac{1}{x}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

10. Let $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ -\dfrac{2}{1+x}, & x > 1 \end{cases}$.

Show that $\underset{x \to 1}{\text{Lt}} f(x)$ does not exist.

11. Using definition of limit, show that

(*i*) $\underset{x \to 0}{\text{Lt}}\ x \sin \dfrac{1}{x} = 0$ (*ii*) $\underset{x \to 0}{\text{Lt}}\ x \cos \dfrac{1}{x} = 0$

(*iii*) $\underset{x \to 0}{\text{Lt}}\ x^2 \sin \dfrac{1}{x} = 0$ (*iv*) $\underset{x \to 0}{\text{Lt}}\ x^2 \cos \dfrac{1}{x} = 0.$

Art-9. Continuity of a Function

We know that a function is said to be continuous at a point if it has no break at that point. This is possible if the values of the function in neighbourhood of that point are very-very close to the value of the function at that point. Hence continuity of the function at a point is closely related to the limiting behaviour of the function at the point under consideration. We given below the definition of continuity in precise form.

Right and left continuity of a function

Def. A real valued function $f(x)$ is said to be left-continuos or continuous from the left iff given $\varepsilon > 0$, however small, there exists a + ve real number δ (ε) such that

$$|f(x) - f(a)| < \varepsilon \text{ for } a - \delta < x < a$$

In symbols, we can say, f is left continuous at a

iff $\underset{x \to a-}{\text{Lt}} f(x) = f(a)$

Similarly a real valued function $f(x)$ is said to be right continuous or continuous from right at a iff given $\varepsilon > 0$, however small, there exists a + ve real number δ (ε) such that

$$|f(x) - f(a)| < \varepsilon \text{ for } a < x < a + \delta.$$

In symbols, f is right continuous at a iff $\underset{x \to a+}{\text{Lt}} f(x) = f(a)$

Continuity of a function at a point

A function f is said to be continuous at $x = a$ if given $\varepsilon > 0$, however small, there exists a + ve real number δ (ε) such that

$$|f(x) - f(a)| < \varepsilon \text{ for } |x - a| < \delta$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

In symbols, f is said to be continuous at a

iff $\underset{x \to a}{\text{Lt}} f(x) = f(a)$

Now $\underset{x \to a}{\text{Lt}} f(x)$ exists

iff $\underset{x \to a-}{\text{Lt}} f(x) = \underset{x \to a+}{\text{Lt}} f(x)$ (both existing finitely)

Hence $f(x)$ is continuous at a if it is left as well as right continuous at a. A function f which is not continuous at a point is said to be discontinuous at the point.

Note 1. If a function f is not continuous at $x = a$, then f is said to be **discontinuous** at $x = a$.

2. A function f is discontinuous at $x = a$ in the following cases :

(*i*) f is not defined at $x = a$ *i.e.*, $f(a)$ does not exist.

(*ii*) $\underset{x \to a}{\text{Lt}} f(x)$ does not exist. This happens in the following cases :

Case I. $\underset{x \to a-}{\text{Lt}} f(x)$ does not exist.

Case II. $\underset{x \to a+}{\text{Lt}} f(x)$ does not exist.

Case III. $\underset{x \to a-}{\text{Lt}} f(x)$ and $\underset{x \to a+}{\text{Lt}} f(x)$ both exist but are not equal.

(*iii*) $\underset{x \to a}{\text{Lt}} f(x)$ exists but $\underset{x \to a}{\text{Lt}} f(x) \neq f(a)$.

KINDS OF DISCONTINUITY

1. Removable discontinuity

Sometimes a function f is not defined at a or $f(a)$ is defined in such a way that

$$\underset{x \to a}{\text{Lt}} f(x) \neq f(a)$$

Then this discontinuity at a can be removed by defining $f(a)$ in such a way that $\underset{x \to a}{\text{Lt}} f(x) = f(a)$

This kind of discontinuity is known as removable discontinuity.

2. Discontinuity of first kind

If $\underset{x \to a-}{\text{Lt}} f(x)$ or $\underset{x \to a+}{\text{Lt}} f(a)$ both exists finitely but

$\underset{x \to a-}{\text{Lt}} f(x) \neq \underset{x \to a+}{\text{Lt}} f(x)$, then $f(x)$ is said to possess discontinuity of first kind at a or ordinary discontinuity at a.

3. Discontinuity of second kind

If $\underset{x \to a-}{\text{Lt}} f(x)$ or $\underset{x \to a+}{\text{Lt}} f(x)$ does not exist, then $f(x)$ is said to possess discontinuity of second kind at a.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Continuity of a function in an interval

(I) Open-Interval. A real valued function f defined on the open interval (a, b) is said to be continuous on (a, b) iff f is continuous at $x = c$ for all $c \in (a, b)$.

(II) Closed-Interval. A real valued function f defined on the closed interval $[a, b]$ is said to be continuous on $[a, b]$ iff

(*i*) f is right continuous at a *i.e.*, $\underset{x \to a+}{\text{Lt}} f(x) = f(a)$

(*ii*) f is left continuous at b *i.e.*, $\underset{x \to b-}{\text{Lt}} f(x) = f(b)$

(*iii*) f is continuous at c $\forall\ c \in (a, b)$ *i.e.*, $\underset{x \to c}{\text{Lt}} f(x) = f(c)$ $\forall\ c \in (a, b)$.

Remark. Such a function has continuous graph on $[a, b]$.

Continuous function. If a function f is continuous at every point of its domain, it is said to be a continuous function.

Domain of continuity. The set of all points where the function is continuous is called the domain of continuity.

Art-10. If f, g are two continuous functions at a, then

(*i*) kf is continuous at a, $\forall\ k \in \mathbf{R}$

(*ii*) $(f \pm g)$ is continuous at a

(*iii*) fg if continuous at a

(*iv*) $\dfrac{f}{g}$ is continuous at a, provided $g(a) \neq 0$.

Proof. $\because$ f, g are continuous at $x = a$

$\therefore$ $\underset{x \to a}{\text{Lt}} f(x) = f(a)$ and $\underset{x \to a}{\text{Lt}} g(x) = g(a)$

(*i*) $\underset{x \to a}{\text{Lt}} (kf)(x) = k \underset{x \to a}{\text{Lt}} f(x) = kf(a) = (kf)(a)$

$\Rightarrow$ kf is continuous at $x = a$ $\forall\ k \in \mathbf{R}$

(*ii*) $\underset{x \to a}{\text{Lt}} (f \pm g)(x) = \underset{x \to a}{\text{Lt}} [f(x) \pm g(x)] = \underset{x \to a}{\text{Lt}} f(x) \pm \underset{x \to a}{\text{Lt}} g(x)$

$$= f(a) \pm g(a) = (f \pm g)\, a$$

$\Rightarrow$ $(f \pm g)$ is continuous at $x = a$

(*iii*) $\underset{x \to a}{\text{Lt}} (fg)(x) = \underset{x \to a}{\text{Lt}} [f(x)\, g(x)] = \underset{x \to a}{\text{Lt}} f(x) \underset{x \to a}{\text{Lt}} g(x)$

$$= f(a)\, g(a) = (fg)(a)$$

$\Rightarrow$ fg is continuous at $x = a$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

(*iv*) $\underset{x \to a}{\text{Lt}} \left(\frac{f}{g}\right)(x) = \underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)} = \frac{\underset{x \to a}{\text{Lt}} f(x)}{\underset{x \to a}{\text{Lt}} g(x)} = \frac{f(a)}{g(a)} = \left(\frac{f}{g}\right)(a)$

$\Rightarrow \quad \frac{f}{g}$ is continuous at $x = a$, provided $g(a) \neq 0$.

Cor. If f is continuous at $x = a$, then $\frac{1}{f}$ is also continuous at $x = a$, provided $f(a) \neq 0$

Art-11. Prove that

(*i*) a constant function is continuous everywhere,

(*ii*) a real polynomial function is continuous everywhere,

(*iii*) a rational is continuous at every point of its domain.

Proof. (*i*) Let f be a constant function.

$\therefore \quad f(x) = k \quad \forall \; x \in \mathbf{R}$. Let a be any real number.

Now $\underset{x \to a}{\text{Lt}} f(x) = \underset{x \to a}{\text{Lt}} (k) = k$. Also $f(a) = k$

$\therefore \quad \underset{x \to a}{\text{Lt}} f(x) = f(a) \quad \forall \; a \in \mathbf{R} \quad \Rightarrow \quad f$ is continuous everywhere.

(*ii*) Let $f(x) = p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n$ be a real polynomial function and a be any real number.

$$\therefore \quad \underset{x \to a}{\text{Lt}} f(x) = \underset{x \to a}{\text{Lt}} [p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n]$$

$$= p_0 \underset{x \to a}{\text{Lt}} x^n + p_1 \underset{x \to a}{\text{Lt}} x^{n-1} + p_2 \underset{x \to a}{\text{Lt}} x^{n-2} + \ldots + \underset{x \to a}{\text{Lt}} p_n$$

$$= p_0 a^n + p_1 a^{n-1} + p_2 a^{n-2} + \ldots + p_n$$

$$\therefore \quad \underset{x \to a}{\text{Lt}} f(x) = f(a) \quad \forall \; a \in \mathbf{R}$$

$\therefore \quad f$ is continuous everywhere.

(*iii*) Let $f(x) = \frac{g(x)}{h(x)}$ be a rational function, so that $g(x)$, $h(x)$ are polynomial function. Domain of f will be set of all real numbers excluding those where $g(x) = 0$. Let a be any member of D_f, then $h(a) \neq 0$.

$$\underset{x \to a}{\text{Lt}} f(x) = \underset{x \to a}{\text{Lt}} \frac{g(x)}{h(x)} = \frac{\underset{x \to a}{\text{Lt}} g(x)}{\underset{x \to a}{\text{Lt}} h(x)} = \frac{g(a)}{h(a)} = f(a)$$

$\therefore \quad f$ is continuous at $x = a$

$\Rightarrow \quad f$ is continuous at every point of its domain.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Note. If $f+g$, $f-g$, fg or $\dfrac{f}{g}$ is a continuous function at $x=a$ then f and g need not be continuous at $x=a$.

e.g. Let $f(x)=\begin{cases}-1, & x\geq 0\\ 1, & x<0\end{cases}$

$$g(x)=\begin{cases}1, & x\geq 0\\ -1, & x<0\end{cases}$$

$\therefore\ (f+g)(x)=0\ \ \forall\ x\in\mathbf{R}$, and $(fg)(x)=-1\ \ \forall\ x\in\mathbf{R}$

Now $f+g$ and fg are constant functions and therefore are continuous at $x=0$, but f and g are not continuous at $x=0$.

Art-12. Let D_1, D_2 be domain of continuity of functions f and g such that $D_1\cap D_2\neq\phi$. Then

(*i*) kf is continuous on D_1

(*ii*) $f\pm g$ is continuous on $D_1\cap D_2$

(*iii*) fg is continuous on $D_1\cap D_2$

(*iv*) $\dfrac{f}{g}$ is continuous on $D_1\cap D_2$ leaving those points where $g(x)=0$.

Proof. Students can prove it easily.

Art-13. (*a*) Absolute function

The absolute function of the function f, denoted by $|f|$ is defined by

$$|f|(x)=|f(x)|\quad\forall\ x\in D_f.$$

Art-13. (*b*) If f is continuous at $x=a$, then $|f|$ is also continuous at $x=a$.

Proof. $\because$ f is continuous at $x=a$

$\therefore$ given $\varepsilon>0$, however small, we can find a positive real number $\delta(\varepsilon)$ such that

$$|f(x)-f(a)|<\varepsilon \text{ for } |x-a|<\delta \qquad \ldots(1)$$

Now $\big||f|(x)-|f|(a)\big| = \big||f(x)|-|f(a)|\big|$

$\leq |f(x)-f(a)|$ $\qquad [\because \big||a|-|b|\big|\leq|a-b|]$

$<\varepsilon$ $\qquad [\because \text{ of } (1)]$

$\therefore\ \big||f|(x)-|f|(a)\big|<\varepsilon$ for $|x-a|<\delta$

$|f|$ is continuous at $x=a$.

Note. The converse of the above theorem is not true. In other words if $|f|$ is continuous at $x=a$, then f is not necessarily continuous at $x=a$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 1. Lt $f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$

$\therefore \quad |f|(x) = 1 \quad \forall\ x \in \mathbf{R}$

$\Rightarrow \quad |f|$ is a constant function

$\Rightarrow \quad |f|$ is continuous at $x = 0$

But $\underset{x \to 0+}{\text{Lt}}\ f(x) = 1$ and $\underset{x \to 0-}{\text{Lt}}\ f(x) = -1$

$\Rightarrow \quad \underset{x \to 0}{\text{Lt}}\ f(x)$ does not exist

$\Rightarrow \quad f$ is not continuous at $x = 0$

$\therefore \quad |f|$ is continuous at $x = 0$, but f is not continuous at $x = 0$.

Example 2. If $f(x) = \begin{cases} -1, & \text{for } x \leq 1 \\ 1, & \text{for } x > 1 \end{cases}$

then show that $|f(x)|$ is continuous $\forall\, x \in \mathbf{R}$.

Sol. We have

$$f(x) = \begin{cases} -1, & \text{for } x \leq 1 \\ 1, & \text{for } x > 1 \end{cases}$$

$\therefore |f(x)| = 1\ \forall\, x \in \mathbf{R}$

$\Rightarrow |f|$ is a constant function.

$\Rightarrow |f|$ is continuous $\forall\, x \in \mathbf{R}$. [$\because$ a constant function is continuous everywhere]

ILLUSTRATIVE EXAMPLES

Example 1. Is the function $f(x) = \begin{cases} n-1, & x = 1 \\ \dfrac{1-x^n}{1-x}, & x \neq 1,\ n \in \mathbf{N} \end{cases}$

continuous at $x = 1$?

If discontinuous, then state the kind of discontinuity.

Sol. Here $f(x) = \begin{cases} n-1, & x = 1 \\ \dfrac{1-x^n}{1-x}, & x \neq 1,\ n \in \mathbf{N} \end{cases}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\underset{x \to 1}{\text{Lt}} f(x) = \underset{x \to 1}{\text{Lt}} \frac{1-x^n}{1-x} = \underset{x \to 1}{\text{Lt}} \frac{(1-x)(1+x+x^2+\ldots+x^{n-1})}{(1-x)}$$

$$= \underset{x \to 1}{\text{Lt}} (1+x+x^2+\cdot+x^{n-1}) = 1+1+1+\ldots+1$$

$$= 1+1+1+\ldots \text{ upto } n \text{ terms}$$

$$= n$$

Also $f(1) = n-1$

$\therefore \underset{x \to 1}{\text{Lt}} f(x) \neq f(1)$

$\therefore$ f is not continuous at $x = 1$

Here f has removable discontinuity.

Example 2. Examine the continuity of the function :

$$f(x) = \begin{cases} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x} & , \quad x \neq 0 \\ \dfrac{1}{5} & , \quad x = 0 \end{cases}$$

at $x = 0$

Sol. $$f(x) = \begin{cases} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x} & , \quad x \neq 0 \\ \dfrac{1}{5} & , \quad x = 0 \end{cases}$$

$$\underset{x \to 0}{\text{Lt}} f(x) = \underset{x \to 0}{\text{Lt}} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \underset{x \to 0}{\text{Lt}} \frac{\sqrt{1+x}-\sqrt{1+x}}{x} \times \frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{(1+x)-(1-x)}{x(\sqrt{1+x}+\sqrt{1-x})} = \underset{x \to 0}{\text{Lt}} \frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2}{\sqrt{1+x}+\sqrt{1-x}} = \frac{2}{\sqrt{1+0}+\sqrt{1-0}} = \frac{2}{1+1} = \frac{2}{2} = 1$$

Also $f(0) = \dfrac{1}{5}$

$\therefore \underset{x \to 0}{\text{Lt}} f(x) \neq f(0)$

$\therefore$ f is discontinuous at $x = 0$.

Example 3. Examine the continuity of the function

$$f(x) = \begin{cases} \dfrac{\sqrt{1-\cos 2x}}{x}, & x \neq 0 \\ 0 \quad , & x = 0 \end{cases}$$

at $x = 0$.

In case of discontinuity, state the kind of discontinuity.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Sol. $$f(x) = \begin{cases} \dfrac{\sqrt{1-\cos 2x}}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$= \begin{cases} \dfrac{\sqrt{2\sin^2 x}}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$= \begin{cases} \dfrac{\sqrt{2}\,|\sin x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\underset{x \to 0-}{\text{Lt}} f(x) = \underset{x \to 0-}{\text{Lt}} \frac{\sqrt{2}\,|\sin x|}{x}$$

[Put $x = 0 - h$, $h > 0$ so that $h \to 0$ as $x \to 0-$]

$$= \underset{h \to 0}{\text{Lt}} \frac{\sqrt{2}\,|\sin(-h)|}{-h} = \underset{h \to 0}{\text{Lt}} \frac{\sqrt{2}\sin h}{-h}$$

$$= -\sqrt{2} \underset{h \to 0}{\text{Lt}} \frac{\sin h}{h} = -\sqrt{2} \times 1 = -\sqrt{2}$$

$$\underset{x \to 0+}{\text{Lt}} f(x) = \underset{x \to 0+}{\text{Lt}} \frac{\sqrt{2}\,|\sin x|}{x}$$

[Put $x = 0 + h$, $h > 0$ so that $h \to 0$ as $x \to 0+$]

$$= \underset{h \to 0}{\text{Lt}} \frac{\sqrt{2}\,|\sin h|}{h} = \sqrt{2} \underset{h \to 0}{\text{Lt}} \frac{\sin h}{h} = \sqrt{2} \times 1 = \sqrt{2}$$

$\therefore \quad \underset{x \to 0-}{\text{Lt}} f(x) \neq \underset{x \to 0+}{\text{Lt}} f(x)$

$\therefore \quad \underset{x \to 0}{\text{Lt}} f(x)$ does not exist

$\therefore \quad f$ is discontinuous at $x = 0$ and discontinuity is of first kind.

Example 4. Discuss the continuity at $x = 0$ of the function

$$f(x) = \begin{cases} \sin^{-1}|x|, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Sol. Here $f(x) = \begin{cases} \sin^{-1}|x|, & x \neq 0 \\ 0, & x = 0 \end{cases}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\underset{x \to 0-}{\text{Lt}} f(x) = \underset{x \to 0-}{\text{Lt}} \sin^{-1}|x| = \underset{x \to 0-}{\text{Lt}} \sin^{-1}(-x) = \underset{x \to 0-}{\text{Lt}} (-\sin^{-1} x)$$

$$= \underset{t \to 0-}{\text{Lt}} (-t)$$

[Put $\sin^{-1} x = t$ or $x = \sin t$ so that $t \to 0-$ as $x \to 0-$]

$$= \underset{h \to 0}{\text{Lt}} \{-(0-h)\}$$ [Put $t = 0 - h,\ h > 0$]

$$= \underset{h \to 0}{\text{Lt}} h = 0$$

$$\therefore \underset{x \to 0-}{\text{Lt}} f(x) = 0$$

$$\underset{x \to 0+}{\text{Lt}} f(x) = \underset{x \to 0+}{\text{Lt}} \sin^{-1}|x| = \underset{x \to 0+}{\text{Lt}} \sin^{-1} x = \underset{t \to 0+}{\text{Lt}} (t)$$

[Put $\sin^{-1} x = t$ or $x = \sin t$ so that $t \to 0+$ as $x \to 0+$]

$$= \underset{h \to 0}{\text{Lt}} (0+h)$$ [Put $t = 0 + h,\ h > 0$]

$$= \underset{h \to 0}{\text{Lt}} h = 0$$

$$\therefore \underset{x \to 0+}{\text{Lt}} f(x) = 0$$

Also $f(0) = 0$

$$\therefore \underset{x \to 0-}{\text{Lt}} f(x) = \underset{x \to 0+}{\text{Lt}} f(x) = f(0)$$

$\therefore$ f is continuous at $x = 0$.

Example 5. Examine the continuity of

$$f(x) = \begin{cases} x \sin \dfrac{1}{x} - 1, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

at $x = 0$. If discontinuous, give the nature discontinuity.

Sol. Here $f(x) = \begin{cases} x \sin \dfrac{1}{x} - 1, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$

Let $g(x) = x,\ h(x) = \sin \dfrac{1}{x}$

Now $\underset{x \to 0}{\text{Lt}} g(x) = \underset{x \to 0}{\text{Lt}} x = 0$

Also $|h(x)| = \left|\sin \dfrac{1}{x}\right| \leq 1 \quad \forall\ x \in \mathbf{R},\ x \neq 0$

$\therefore$ $h(x)$ is bounded in a deleted neighbourhood of 0.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$\therefore \quad \underset{x \to 0}{\text{Lt}} \; g(x)\, h(x) = 0 \quad \Rightarrow \quad \underset{x \to 0}{\text{Lt}} \; x \sin \dfrac{1}{x} = 0$

$\therefore \quad \underset{x \to 0}{\text{Lt}} \left(x \sin \dfrac{1}{x} - 1 \right) = 0 - 1 = -1$

$\Rightarrow \quad \underset{x \to 0}{\text{Lt}} \; f(x) = -1$. Also $f(0) = 0$

$\therefore \quad \underset{x \to 0}{\text{Lt}} \; f(x) \neq f(0)$

$\Rightarrow$ f is discontinuous at $x = 0$.

Discontinuity is removable as function becomes continuous if $f(0) = -1$.

Example 6. Examine the continuity of the function

$$f(x) = \begin{cases} 1 + x, & x \leq 2 \\ 5 - x, & x > 2 \end{cases}$$

at $x = 2$.

Sol. Here $f(x) = \begin{cases} 1 + x, & x \leq 2 \\ 5 - x, & x > 2 \end{cases}$]

$$\underset{x \to 2-}{\text{Lt}} \; f(x) = \underset{x \to 2-}{\text{Lt}} \; (1 + x) = \underset{h \to 0}{\text{Lt}} \; [1 + (2 - h)] \qquad [\text{Put } x = 2 - h]$$

$$= \underset{h \to 0}{\text{Lt}} \; (3 - h) = 3 - 0 = 3$$

$$\underset{x \to 2+}{\text{Lt}} \; f(x) = \underset{x \to 2+}{\text{Lt}} \; (5 - x) = \underset{h \to 0}{\text{Lt}} \; [5 - (2 + h)] \qquad [\text{Put } x = 2 + h]$$

$$= \underset{h \to 0}{\text{Lt}} \; (3 - h) = 3 - 0 = 3.$$

Now $\quad f(x) = 1 + x$, for $x \leq 2$

$\therefore \quad f(2) = 1 + 2 = 3$

$\therefore \quad \underset{x \to 2-}{\text{Lt}} \; f(x) = \underset{x \to 2+}{\text{Lt}} \; f(x) = f(2)$

$\therefore \quad f$ is continuous at $x = 2$.

Example 7. Discuss continuity of $f(x) = \begin{cases} \dfrac{|x| + x}{3}, & x \leq 3 \\ \dfrac{2|x - 3|}{x - 3}, & x > 3 \end{cases}$

over **R**.

Sol. We have

$$f(x) = \begin{cases} \dfrac{|x| + x}{3}, & x \leq 3 \\ \dfrac{2|x - 3|}{x - 3}, & x > 3 \end{cases}$$

Function is defined at all points of the domain. Only possible point of discontinuity is $x = 3$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Now $\underset{x \to 3-}{Lt} f(x) = \underset{x \to 3-}{Lt} \dfrac{|x| + x}{3}$ [Put $x = 3 - h$, $h > 0$ so that $h \to 0$ as $x \to 3 -$]

$$= \underset{x \to 0}{Lt} \frac{|3 - h| + (3 - h)}{3}$$

$$= \frac{|3 - 0| + (3 - 0)}{3} = \frac{3 + 3}{3} = \frac{6}{3} = 2$$

$\underset{x \to 3+}{Lt} f(x) = \underset{x \to 3+}{Lt} \dfrac{2|x - 3}{x - 3}$ [Put $x = 3 + h$, $h > 0$ so that $h \to 0$ as $x \to 3 +$]

$$= \underset{x \to 0}{Lt} \frac{2|3 + h - 3|}{3 + h - 3}$$

$$= \underset{x \to 0}{Lt} \frac{2h}{h} = 2$$

Also $f(3) = \dfrac{|3| + 3}{3} = \dfrac{3 + 3}{3} = \dfrac{1}{3} = 2$

$\therefore \underset{x \to 3-}{Lt} f(x) = \underset{x \to 3+}{Lt} f(x) = f(3)$

$\therefore f$ is continuous at $x = 3$.

$\therefore f$ is continuous over **R**.

Example 8. Show that the function $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

is discontinuous at $x = 0$.

Sol. Here $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$\underset{x \to 0-}{\text{Lt}} f(x) = \underset{x \to 0}{\text{Lt}} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$ [Put $x = 0 - h$, $h > 0$ so that $h \to 0$ as $x \to 0 -$]

$$= \underset{h \to 0}{\text{Lt}} \frac{e^{-\frac{1}{h}} - 1}{e^{-\frac{1}{h}} + 1} = \frac{0 - 1}{0 + 1} \qquad \left[\because e^{-\frac{1}{h}} \to 0 \text{ as } h \to 0\right]$$

$$= -1$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\underset{x \to 0+}{\text{Lt}} f(x) = \underset{x \to 0+}{\text{Lt}} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$$

[Put $x = 0 + h$, $h > 0$ so that $h \to 0$ as $x \to 0+$]

$$= \underset{h \to 0}{\text{Lt}} \frac{e^{\frac{1}{h}} - 1}{e^{\frac{1}{h}} + 1} = \underset{h \to 0}{\text{Lt}} \frac{\dfrac{e^{\frac{1}{h}}}{e^{\frac{1}{h}}} - \dfrac{1}{e^{\frac{1}{h}}}}{\dfrac{e^{\frac{1}{h}}}{e^{\frac{1}{h}}} + \dfrac{1}{e^{\frac{1}{h}}}} = \underset{h \to 0}{\text{Lt}} \frac{1 - e^{-\frac{1}{h}}}{1 + e^{-\frac{1}{h}}}$$

$$= \frac{1-0}{1+0} \qquad \left[\because e^{-\frac{1}{h}} \to 0 \text{ as } h \to 0 \right]$$

$\therefore \quad \underset{x \to 0-}{\text{Lt}} f(x) \neq \underset{x \to 0+}{\text{Lt}} f(x)$

$\Rightarrow \quad \underset{x \to 0}{\text{Lt}} f(x)$ does not exist

$\Rightarrow \quad f(x)$ is discontinuous at $x = 0$.

Example 9. Find all points at which the greatest integer function $[x]$ is continuous.

Sol. Let $f(x) = [x]$. $D_f = \mathbf{R}$

Let a be any real number $\in D_f$.

Two cases arise :

Case I. If a is not an integer, then

$$\underset{x \to a}{\text{Lt}} f(x) = \underset{x \to a}{\text{Lt}} [x] = [a] = f(a)$$

$\Rightarrow \quad f$ is continuous at $x = a$

Case II. If $a \in I$, then $f(a) = [a] = a$ and

$$\underset{x \to a-}{\text{Lt}} f(x) = \underset{x \to a-}{\text{Lt}} [x] = a - 1$$

$$\underset{x \to a+}{\text{Lt}} f(x) = \underset{x \to a+}{\text{Lt}} [x] = a$$

$\therefore \quad \underset{x \to a-}{\text{Lt}} f(x) \neq \underset{x \to a+}{\text{Lt}} f(x)$

$\therefore \quad f$ is not continuous at $x = a$, $a \in I$.

$\therefore$ function is discontinuous at every integral point.

Domain of continuity for the function $[x]$ is $\mathbf{R} - \mathbf{I}$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 10. If $[x]$ denotes greatest integer $\leq x$, then show that

$$f(x) = \begin{cases} [x]+[-x], & x \neq 0 \\ 0 \quad , & x = 0 \end{cases}$$

is discontinuous at $x = 0$.

Also write the kind of discontinuity.

Sol. We have

$$f(x) = \begin{cases} [x]+[-x], & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$\underset{x \to 0-}{\text{Lt}} f(x) = \underset{x \to 0-}{\text{Lt}} \{[x]+[-x]\}$$

[Put $x = 0 - h, h > 0$ so that $h \to 0$ as $x \to 0-$]

$$= \underset{h \to 0}{\text{Lt}} \{[0-h]+[0+h]\} = -1+0 = -1$$

$$\underset{x \to 0+}{\text{Lt}} f(x) = \underset{x \to 0+}{\text{Lt}} \{[x]+[-x]\}$$

[Put $x = 0 + h, h > 0$ so that $h \to 0$ as $x \to 0+$]

$$= \underset{h \to 0}{\text{Lt}} \{[0+h]+[-0-h]\} = \underset{h \to 0}{\text{Lt}} \{[h]+[-h]\} = 0+(-1) = -1$$

Also $f(0) = 0$

$$\therefore \underset{x \to 0-}{\text{Lt}} f(x) = \underset{x \to 0+}{\text{Lt}} f(x) \neq f(0)$$

$\therefore$ $f(x)$ is discontinuous at $x = 0$.

Example 11. Prove that the function $f(x) = \begin{cases} \dfrac{x}{|x| - 2x}, & x \neq 0 \\ k, & x = 0 \end{cases}$

remains discontinuous at $x = 0$, regardless of the choice of k.

Sol. Here $f(x) = \begin{cases} \dfrac{x}{|x| - 2x}, & x \neq 0 \\ k, & x = 0 \end{cases}$

$$\underset{x \to 0-}{\text{Lt}} f(x) = \underset{x \to 0-}{\text{Lt}} \frac{x}{|x| - 2x}$$

[Put $x = 0 - h, h > 0$ so that $h \to 0$ as $x \to 0-$]

$$= \underset{h \to 0}{\text{Lt}} \frac{0-h}{|0-h| - 2(0-h)}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{-h}{h+2h} = \underset{h \to 0}{\text{Lt}} \frac{-1}{1+2} = -\frac{1}{3}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\underset{x\to 0+}{\text{Lt}} f(x) = \underset{x\to 0+}{\text{Lt}} \frac{x}{|x|-2x} \qquad [\text{Put } x = 0+h,\ h>0 \text{ so that } h\to 0 \text{ as } x\to 0+]$$

$$= \underset{h\to 0}{\text{Lt}} \frac{|0+h|}{|0+h|-2(0+h)}$$

$$= \underset{h\to 0}{\text{Lt}} \frac{h}{h-2h} = \underset{h\to 0}{\text{Lt}} \frac{1}{1-2} = -1$$

$\therefore \quad \underset{x\to 0-}{\text{Lt}} f(x) \neq \underset{x\to 0+}{\text{Lt}} f(x)$

$\Rightarrow \quad \underset{x\to 0}{\text{Lt}} f(x)$ does not exist.

$\Rightarrow \quad f(x)$ is discontinuous at $x = 0$, whatever k may be.

Example 12. Determine the value of constant A so that the function

$$f(x) = \begin{cases} \dfrac{1-\cos x}{x^2}, & x \neq 0 \\ \text{A}, & x = 0 \end{cases}$$

is continuous at $x = 0$.

Sol. Here $f(x) = \begin{cases} \dfrac{1-\cos x}{x^2}, & x \neq 0 \\ \text{A}, & x = 0 \end{cases}$

$$\underset{x\to 0}{\text{Lt}} f(x) = \underset{x\to 0}{\text{Lt}} \frac{1-\cos x}{x^2} = \underset{x\to 0}{\text{Lt}} \frac{2\sin^2\frac{x}{2}}{x^2} = \frac{1}{2}\underset{x\to 0}{\text{Lt}} \left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{2}\times(1)^2 = \frac{1}{2}$$

Since $f(x)$ is continuous at $x = 0$.

$\therefore \quad f(0) = \underset{x\to 0}{\text{Lt}} f(x) \qquad \Rightarrow \qquad \text{A} = \dfrac{1}{2}.$

Example 13. Let $f(x) = \begin{cases} 1, & x \leq 3 \\ ax+b, & 3 < x < 5 \\ 7, & 5 \leq x \end{cases}$.

Determine the constants a and b so that f may be continuous for all x.

Sol. Here $f(x) = \begin{cases} 1, & x \leq 3 \\ ax+b, & 3 < x < 5 \\ 7, & 5 \leq x \end{cases}$

$\because$ f is continuous for all x

$\therefore$ f is continuous at $x = 3, 5$

$\therefore \quad \underset{x\to 3+}{\text{Lt}} f(x) = f(3)$ and $\underset{x\to 5-}{\text{Lt}} f(x) = f(5)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Now $\underset{x \to 3+}{\text{Lt}} f(x) = f(3) \quad \Rightarrow \quad \underset{x \to 3+}{\text{Lt}} (a x + b) = 1$

$\Rightarrow \quad \underset{h \to 0}{\text{Lt}} [3(a + h) + b] = 1$ (Put $x = 3 + h,\ h > 0$)

$\Rightarrow \quad 3a + b = 1$ ...(1)

Also $\underset{x \to 5-}{\text{Lt}} f(x) = f(5) \quad \Rightarrow \quad \underset{x \to 5-}{\text{Lt}} (a x + b) = 7$

$\Rightarrow \quad \underset{h \to 0}{\text{Lt}} [a(5 - h) + b] = 7$ (Put $x = 5 - h,\ h > 0$)

$\Rightarrow \quad 5a + b = 7$ ...(2)

Subtracting (1) from (2), we get,

$2a = 6 \Rightarrow a = 3$

Putting this value of a in (1), we get,

$9 + b = 1 \Rightarrow b = -8$

$\therefore$ when $a = 3,\ b = -8,\ f$ is continuous at all points.

Example 14. For what values of a and b will the following function be continuous $\forall x$?

$$f(x) = \begin{cases} 13, & x \le 2 \\ a x^2 + b x + 1, & 2 < x < 3 \\ 17 - a x, & x \ge 3 \end{cases}$$

(J.U. 2016)

Sol. We have

$$f(x) = \begin{cases} 13, & \text{if } x \le 2 \\ a x^2 + b x + 1, & \text{if } 2 < x < 3 \\ 17 - a x, & \text{if } x \ge 3 \end{cases}$$

Since $f(x)$ is continuous for all x

$\therefore$ $f(x)$ is continuous at $x = 2, 3$

Since $f(x)$ is continuous at $x = 2$

$\therefore \quad \underset{x \to 2-}{\text{Lt}} f(x) = \underset{x \to 2+}{\text{Lt}} f(x)$

$\Rightarrow \quad \underset{x \to 2-}{\text{Lt}} (13) = \underset{x \to 2+}{\text{Lt}} (a x^2 + b x + 1)$

$\Rightarrow \quad 13 = 4a + 2b + 1 \Rightarrow 4a + 2b - 12 = 0$

$\Rightarrow \quad 2a + b - 6 = 0$ ...(1)

Again as $f(x)$ is continuous at $x = 3$

$\therefore \quad \underset{x \to 3-}{\text{Lt}} f(x) = \underset{x \to 3+}{\text{Lt}} f(x)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\therefore \quad \underset{x \to 3-}{\text{Lt}} (ax^2 + bx + 1) = \underset{x \to 3+}{\text{Lt}} (17 - ax)$$

$$\Rightarrow \quad 9a + 3b + 1 = 17 - 3a$$

$$\Rightarrow \quad 12a + 3b - 16 = 0 \qquad \ldots(2)$$

Solving (1) from (2), we get,

$$\frac{a}{-16+18} = \frac{b}{-72+32} = \frac{1}{6-12} \quad \text{or} \quad \frac{a}{2} = \frac{b}{-40} = \frac{1}{-6}$$

$$\therefore \quad \frac{a}{1} = \frac{b}{-20} = \frac{1}{-3}$$

$$\therefore \quad a = -\frac{1}{3}, \quad b = \frac{20}{3}$$

Example 15. Find the domain of continuity of $f(x) = [x]$.

Sol. Here $f(x) = [x]$. Therefore $D_f = \mathbf{R}$

Let a be any real number. Then either a is an integer or not an integer.

Case I. If a is an integer, then

$$\underset{x \to a-}{\text{Lt}} f(x) = \underset{x \to a-}{\text{Lt}} [x] = a - 1 \quad \text{and} \quad \underset{x \to a+}{\text{Lt}} f(x) = \underset{x \to a+}{\text{Lt}} [x] = a$$

$$\therefore \quad \underset{x \to a-}{\text{Lt}} f(x) \neq \underset{x \to a+}{\text{Lt}} f(x)$$

$\therefore$ f is discontinuous at $x = a$

$\Rightarrow$ f is discontinuous at integral points.

Case II. If a is not an integer, then

$$\underset{x \to a}{\text{Lt}} f(x) = \underset{x \to a}{\text{Lt}} [x] = [a] = f(a)$$

$\therefore$ f is continuous at all non-integral real points.

From cases I and II, it is clear that domain of continuity of f is the set **R** of all reals other than integers.

Example 16. Find the domain of continuity of the following functions :

$$(i) \quad \frac{\log(1+x)}{[x-1]} \qquad (ii) \quad \frac{\log x}{\sqrt{1-4x^2}} \qquad (iii) \quad \frac{\sqrt{|x|-x}}{\cos^{-1} x}$$

Sol. (*i*) Let $f(x) = \dfrac{\log(1+x)}{[x-1]}$

Put $g(x) = \log(1+x)$, $h(x) = [x-1]$

Now g is continuous for all x in $(-1, \infty)$

$\therefore$ domain of continuity of g is $(-1, \infty)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Again h is continuous for all $x \in \mathbf{R}$ other than integer *i.e.*, h is continuous for all x in $\mathbf{R} - \mathbf{I}$.

Also $h(x) = 0$ when $0 \le x - 1 < 1$ *i.e.*, $1 \le x < 2$

$\therefore$ domain of continuity of $f(x) = \dfrac{g(x)}{h(x)}$ is $\{(-1, \infty) \cap (\mathbf{R} - \mathbf{I})\} - [1, 2)$

(*ii*) Let $f(x) = \dfrac{\log x}{\sqrt{1 - 4x^2}}$

Now $\log x$ is continuous for all $x > 0$ and $\sqrt{1 - 4x^2}$ is continuous for all x satisfying

$$1 - 4x^2 \ge 0$$

i.e., $4x^2 \le 1$

i.e., $x^2 \le \dfrac{1}{4}$

i.e., $|x|^2 \le \dfrac{1}{4}$

i.e., $|x| \le \dfrac{1}{2}$

i.e., $-\dfrac{1}{2} \le x \le \dfrac{1}{2}$. Also $x > 0$ and $\sqrt{1 - 4x^2} = 0$ at $x = -\dfrac{1}{2}, \dfrac{1}{2}$

Hence $D_f = \left(0, \dfrac{1}{2}\right)$

$\because$ $\log x$ and $\sqrt{1 - 4x^2}$ are continuous on $\left(0, \dfrac{1}{2}\right)$

$\therefore$ interval of continuity of $f(x)$ is $\left(0, \dfrac{1}{2}\right)$

(*iii*) For domain of $\sqrt{|x| - x}$ we have

$|x| \ge x \ \forall \ x$

$\sqrt{|x| - x}$ exists $\forall \ x$

$\therefore$ domain of $\sqrt{|x| - x}$ is set of all the reals *i.e.*, domain is $\mathbf{R}$.

Also domain of $\cos^{-1} x$ is $[-1, 1]$

Also $\cos^{-1} x \ne 0$ ($\because$ $\cos^{-1} x$ occurs in the denominator)

i.e., $x = 1$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$\therefore$ domain of $f(x) = \dfrac{\sqrt{|x| - x}}{\cos^{-1} x}$ is $\mathbf{R} \cap [-1, 1] - \{1\}$

$\therefore$ $D_f = [-1, 1)$

$\sqrt{|x| - x}$ and $\cos^{-1} x$ are continuous on $[-1, 1)$

$\therefore$ $f(x) = \dfrac{\sqrt{|x| - x}}{\cos^{-1} x}$ are continuous on $[-1, 1)$

Hence interval of continuity is $[-1, 1)$.

EXERCISE 1 (*b*)

1. Examine the following functions for continuity :

(*i*) $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$ at $x = 1$

(*ii*) $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ at $x = 0$

(*iii*) $f(x) = \begin{cases} \dfrac{2|x| + x^2}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ at $x = 0$

(*iv*) $f(x) = \begin{cases} \dfrac{x - |x|}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$ at $x = 0$

(*v*) $f(x) = \begin{cases} \dfrac{\sin 3x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ at $x = 0$

(*vi*) $f(x) = \begin{cases} \dfrac{1 - \cos x}{x^2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ at $x = 0$

(*vii*) $f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ at $x = 0$

(*viii*) $f(x) = \begin{cases} (x - a) \sin \dfrac{1}{x - a}, & x \neq a \\ 0, & x = a \end{cases}$ at $x = a$

(*ix*) $f(x) = \begin{cases} |x| \cos \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ at $x = 0$

(*x*) $f(x) = \begin{cases} (x - a)^2 \cos \dfrac{1}{x - a}, & x \neq a \\ 1, & x = a \end{cases}$ at $x = a$

2. Examine the following functions for continuity :

(*i*) $f(x) = \begin{cases} \dfrac{1}{x}, & x > 0 \\ x, & x \leq 0 \end{cases}$ at $x = 0$

(*ii*) $f(x) = \begin{cases} -1, & x < 0 \\ 3 - x, & x \geq 0 \end{cases}$ at $x = 0$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

(iii) $f(x) = \begin{cases} 1+x^2, & 0 \le x \le 1 \\ 1-x, & x > 1 \end{cases}$ at $x = 1$

(iv) $f(x) = \begin{cases} \dfrac{2}{5-x}, & x < 3 \\ 5-x, & x \ge 3 \end{cases}$ at $x = 3$

(v) $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ -\dfrac{2}{1+x}, & x > 1 \end{cases}$ at $x = 1$

(vi) $f(x) = \begin{cases} x, & 0 \le x < 1 \\ 2, & x = 1 \\ 2-x, & x > 1 \end{cases}$ at $x = 1$

(vii) $f(x) = \begin{cases} -1+x^2, & x < -1 \\ 1+\dfrac{x+x^2}{2}, & x = -1 \\ 1+x, & x > -1 \end{cases}$ at $x = -1$

(viii) $f(x) = \begin{cases} \dfrac{x^3}{a^2}, & 0 < x < a \\ a, & x = a \\ 2a - \dfrac{a^3}{x^2}, & x > a \end{cases}$ at $x = a$

3. If $f(x) = \begin{cases} e^{\frac{1}{x}}, & x \ne 0 \\ 1, & x = 0 \end{cases}$ find whether f is continuous at $x = 0$.

4. Discuss the continuity of the function

$$f(x) = \begin{cases} \dfrac{3^{\frac{1}{x}}}{1+3^{\frac{1}{x}}}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

at $x = 0$.

5. Discuss the continuity of the function

$$f(x) = \begin{cases} \dfrac{5^{\frac{1}{x}}}{1+5^{\frac{1}{x}}}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

at $x = 0$.

6. Discuss the continuity of the function

$$f(x) = \begin{cases} 5^{\frac{1}{x}}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

at $x = 0$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

7. Examine for continuity of the following functions :

(*i*) $f(x) = \begin{cases} 1, & x \text{ rational} \\ -1, & x \text{ irrational} \end{cases}$ at $x = a \in \mathbf{R}$

(*ii*) $f(x) = \begin{cases} 2 + x, & x \text{ rational} \\ 2{\cdot}5 - x, & x \text{ irrational} \end{cases}$ at $x = {\cdot}25$

(*iii*) $f(x) = \begin{cases} x, & x \text{ rational} \\ -x, & x \text{ irrational} \end{cases}$ at $x = 0$

8. Prove that the function $f(x) = [x]$ is discontinuous, where $[x]$ denotes the greatest integer $\leq x$.

9. Examine the continuity of the function :

$f(x) = [x] + [1 - x]$ at $x = 0$

10. Show that the function f defined by

$$f(x) = \begin{cases} [x-1]+[1-x] & , \quad x \neq 0 \\ 0 & , \quad x = 1 \end{cases}$$

is discontinuous at $x = 1$.

11. Show that the function f defined by

$$f(x) = \begin{cases} [x-2]+[2-x] & , \quad x \neq 2 \\ 0 & , \quad x = 2 \end{cases}$$

is discontinuous at $x = 2$.

12. Discuss the continuity of the function

$$f(x) = \begin{cases} [x]+[-x], & \text{if } x \neq 0 \\ -1 & , \text{if } x = 0 \end{cases}$$

at $x = 0$, where $[x]$ denotes greatest integer $\leq x$.

13. Let $f(x) = \begin{cases} [x]+[-x], & \text{if } x \neq 0 \\ \lambda & , \text{if } x = 0 \end{cases}$

For what value of λ, if any, is f continuous at $x = 0$?

14. (*a*) Is the function $f(x) = \dfrac{\left[\dfrac{1}{3} + x\right] - \dfrac{1}{3}}{x}$ continuous at $x = 2$?

(*b*) Is the function $f(x) = 1 - x + [x] - [1 + x]$ continuous at $x = 0$?

(*c*) Is $f(x) = \begin{cases} \dfrac{[x]-1}{x-1}, & x \neq 1 \\ -1 & , x = 1 \end{cases}$

continuous at $x = 1$?

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

15. Find k for which $f(x)$, defined as

$$f(x) = \begin{cases} x^2 + 2x + k, & x \neq 0 \\ -3, & x = 0 \end{cases}$$

is continuous at $x = 0$.

16. Examine for continuity the function

$$f(x) = \begin{cases} \dfrac{|x-1|}{2(x-1)}, & x \neq 1 \\ k, & x = 1 \end{cases}$$

at $x = 1$.

17. Prove that the function $f(x) = \begin{cases} \dfrac{x}{|x| + 2x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$

remains discontinuous at $x = 0$, regardless of the choice of k.

18. Choose the parameter such that f becomes continuous at indicated points in the following cases :

(*i*) $f(x) = \begin{cases} \dfrac{\sin^2 ax}{x^2}, & x \neq 0 \\ A, & x = 0 \end{cases}$

at $x = 0$ (It being given that $a \neq 0$)

(*ii*) $f(x) = \begin{cases} \dfrac{3x + 4\tan x}{x}, & x \neq 0 \\ A, & x = 0 \end{cases}$

at $x = 0$

(*iii*) $f(x) = \begin{cases} \dfrac{1 - \cos x}{x^2}, & x \neq 0 \\ A, & x = 0 \end{cases}$

at $x = 0$

(*iv*) Let $f(x) = \begin{cases} \dfrac{1 - \cos x}{\sin^2 x}, & x \neq 0 \\ A, & x = 0 \end{cases}$

For what value of A is the function $f(x)$ continuous at $x = 0$?

(*v*) $f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} + 2x^2 + 3, & x \neq 0 \\ A, & x = 0 \end{cases}$

at $x = 0$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

19. If $f(x) = \begin{cases} \dfrac{2|x| + x^2}{3x}, & x \neq 0 \\ A, & x = 0 \end{cases}$.

Show that f is discontinuous at $x = 0$ whatever the value of A may be.

20. Show that the following functions are discontinuous. State clearly, the kind of discontinuity in each case :

(*i*) $f(x) = \begin{cases} \dfrac{|x-2|}{x^2-4}, & x \neq 2 \\ \dfrac{1}{4}, & x = 2 \end{cases}$ at $x = 2$

(*ii*) $f(x) = \begin{cases} [x-1], & x \geq 1 \\ \dfrac{1}{x-1}, & x < 1 \end{cases}$ at $x = 1$

(*iii*) $f(x) = \begin{cases} x^2 - \dfrac{a^2}{2}, & x < a \\ 0, & x = a \\ a^2 - \dfrac{x^2}{2}, & x > a \end{cases}$ at $x = a$

(*iv*) $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & 1 < x < 2 \\ x^3, & 2 \leq x < 3 \\ x+1, & x \geq 3 \end{cases}$ at $x = 2, 3$

21. If f is continuous and g is a discontinuous function then $f + g$ is discontinuous function.

22. Show that the function $f(x) = \begin{cases} |x| \ ; & x \leq 2 \\ [x] \ ; & x > 2 \end{cases}$

is continuous on $[0, 2]$.

23. Let *a* function $f : \mathrm{R} \to \mathrm{R}$ satisfies the equation $f(x+y) = f(x) + f(y)\ \forall\, x, y \in \mathbf{R}$. Then if f is continuous at the point $x = a$, then it is continuous $\forall\, x \in \mathbf{R}$.

24. (*a*) Let $f(x) = \begin{cases} 7 + 2x, & 0 \leq x \leq 2 \\ 13 - x, & 2 < x \leq 4 \end{cases}$, $g(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 3 + x, & 1 < x \leq 2 \end{cases}$

Find the points of continuity of $f + g$ and fg.

(*b*) The functions f and g are defined by

$$f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}, \quad g(x) = \begin{cases} 0, & x \geq 0 \\ 1, & x < 0 \end{cases}$$

Show that the functions f and g are both discontinuous at $x = 0$ but $f + g$ and fg are both continuous at $x = 0$.

(*c*) Give an example of a function which is continuous only at a single point.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

25. Find the value of a and b such that the function defined by

$$f(x) = \begin{cases} 5, & \text{if } x \le 2 \\ ax+b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \ge 10 \end{cases}$$

is a continuous function.

26. If the function $f(x) = \begin{cases} 3ax+b & \text{if } x>1 \\ 11 & \text{if } x=1 \\ 5ax-2b & \text{if } x<1. \end{cases}$

is continuous at $x = 1$, find the values of a and b.

27. For what value of k is the following function continuous at $x = 2$?

$$f(x) = \begin{cases} 2x+1, & x<2 \\ k, & x=2 \\ 3x-1, & x>2 \end{cases}$$

28. Given that $f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & \text{if } x<0 \\ a, & \text{if } x=0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & \text{if } x>0 \end{cases}$

If $f(x)$ is continuous at $x = 0$, find the value of a.

29. Determine the value of a, b, c for which the functions

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x+\sin x}{x}, & x<0 \\ c, & x=0 \\ \dfrac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{\frac{3}{2}}}, & x>0 \end{cases}$$

is continuous at $x = 0$.

30. (*a*) For what choice of a and b are the following functions continuous for all x :

(*i*) $f(x) = \begin{cases} ax^2+b \text{ if } & x>2 \\ 2 \text{ if } & x=2 \\ 2ax-b \text{ if } & x<2 \end{cases}$

(*ii*) $f(x) = \begin{cases} x^2, & x \le 0 \\ ax+b, & x>0 \end{cases}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

(*b*) What conditions should be satisfied by the parameters a and b so that that function

$$f(x) = \begin{cases} x - 1 \ , & x \le 1 \\ a x^2 + b x, & x > 1 \end{cases}$$

is continuous at $x = 1$? Are a, b unique ?

31. Let $f(x) = \begin{cases} x^2 + a & , \ x \ge 0 \\ 2\sqrt{x^2 + 1} + b & , \ x < 0 \end{cases}$

Find a and b so that $f(x)$ is continuous at $x = 0$ and $f(1) = 1$.

32. For what choice of a and b the function

$$f(x) = \begin{cases} 3ax + 2b + 1 & , \ \text{if } x \le 1 \\ 5bx - a + 7 & , \ \text{if } 1 < x < 4 \\ 2ax - bx^2 + 3 & , \ \text{if } x \ge 4 \end{cases}$$

is continuous for all x ?

33. Let $f(x) = \begin{cases} 3 & , \ \text{if } x \le 2 \\ a x^2 + b x + 1 & , \ \text{if } 2 < x < 3 \\ 7 - a x & , \ \text{if } x \ge 3 \end{cases}$.

Determine the constants a and b so that f may be continuous for all x.

34. Let $f(x) = \begin{cases} x^3 - 2x + 3 \ , & x \ge 1 \\ a x^2 + b x & , \ x < 1 \end{cases}$.

Determine the condition satisfied by a and b so that f may be continuous for all a.

Are a and b unique ?

35. Find the domain of continuity of following functions :

(*i*) $\sin x + \log x$ (*ii*) $\sin^{-1} x + \log x$ (*iii*) $\dfrac{\sin^{-1} x}{[x]}$

(*iv*) $x \tan^{-1} x$ (*v*) $e^x + \log x$ (*vi*) $\dfrac{x \cos x}{1 - x^2}$

(*vii*) $\dfrac{e^x + x \cos x}{\sin x}$ (*viii*) $\sin(\cos x)$ (*ix*) $\log x + \sqrt{1 - x^2}$

(*x*) $\sin(\pi [x])$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

ANSWERS

1. (*i*) Discontinuous (*ii*) Discontinuous (*iii*) Discontinuous
(*iv*) Discontinuous (*v*) Discontinuous (*vi*) Discontinuous
(*vii*) Continuous (*viii*) Continuous (*ix*) Continuous
(*x*) Discontinuous

2. (*i*) Discontinuous (*ii*) Discontinuous (*iii*) Discontinuous
(*iv*) Discontinuous (*v*) Discontinuous (*vi*) Discontinuous
(*vii*) Discontinuous (*viii*) Continuous

3. Discontinuous **4.** Discontinuous **5.** Discontinuous

6. Discontinuous

7. (*i*) Discontinuous (*ii*) Continuous (*iii*) Continuous

9. Discontinuous **12.** Continuous

13. $\lambda = -1$

14. (*a*) Continuous (*b*) Continuous (*c*) Discontinuous

15. $k = -3$ **16.** Discontinuous

18. (*i*) $A = a^2$ (*ii*) $A = 7$ (*iii*) $A = \frac{1}{2}$
(*iv*) $A = \frac{1}{2}$ (*v*) $A = 3$

20. (*i*) First kind (*ii*) Second kind (*iii*) Removable
(*iv*) First kind

24. (*a*) Continuous in [0, 2] except at $x = 1$ **25.** $a = 2,\ b = 1$

26. $a = 3,\ b = 2$ **27.** $k = 5$ **28.** $a = 8$

29. $a = -\frac{3}{2},\ c = \frac{1}{2}$, b is any real number except 0

30. (*a*) (*i*) $a = \frac{1}{2},\ b = 0$ (*ii*) $b = 0$ and a may have any value
(*b*) $a + b = 0$, not unique **31.** $a = 0,\ b = -2$

32. $a = \frac{68}{39},\ b = \frac{38}{117}$ **33.** $a = \frac{1}{2},\ b = 0$ **34.** $a + b = 2$, not unique

35. (*i*) $(0, \infty)$ (*ii*) $(0, 1]$ (*iii*) $(-1, 0)$
(*iv*) $(-\infty, \infty)$ (*v*) $(0, \infty)$ (*vi*) $\mathbf{R} - \{-1, 1\}$
(*vii*) $\mathbf{R} - \{n\pi,\ n \in \mathbf{Z}\}$ (*viii*) $\mathbf{R}$ (*ix*) $\mathbf{R}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

2

DIFFERENTIABILITY OF FUNCTIONS

Art-1. Derivative at a Point

Let f be a function with domain D. The derivative of f at the point x of its domain D is defined as the $\underset{h \to 0}{\text{Lt}} \dfrac{f(x+h) - f(x)}{h}$, provided the limit exists and is finite. This derivative (or **differential coefficient**) is denoted by $f'(x)$ or $\dfrac{d}{dx}[f(x)]$

The derivative of f at $c \in$ D, c being an interior point of D, is also defined as $\underset{x \to c}{\text{Lt}} \dfrac{f(x) - f(c)}{x - c}$, provided this limit exists and is finite. This derivative is denoted by $f'(c)$.

Note. 1. If $x \to c - 0$, then derivative is called the **left-handed derivative** or L.H.D., and is denoted by L $f'(c)$ or $f'(c - 0)$.

2. If $x \to c + 0$, then derivative is called the **right-handed derivative** or R.H.D. and is denoted by R $f'(c)$ or $f'(c + 0)$.

3. The derivative at a point exists if L.H.D. and R.H.D. both exist and are equal.

4. A function f is said to be **differentiable** at a point c if and only if the derivative of f exists at c.

5. The process of finding the derivative of a function is called the **differentiation.**

Derivative in an Interval

A function f is said to be derivable in the open interval (a, b) if f is derivable at each point of (a, b).

Further if f is derivable from the right at a and from the left at b, then f is said to be derivable in closed interval $[a, b]$.

Art-2. Geometrical Interpretation of $\dfrac{dy}{dx}$

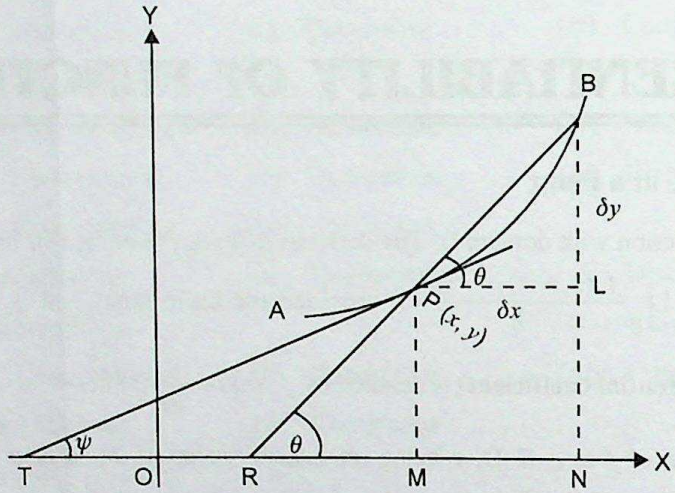
Proof : Take two neighbouring points P(x, y) and Q $(x + \delta x,\ y + \delta y)$ on the graph APQB of the function $y = f(x)$. From P and Q draw $\perp$s PM and QN respectively on OX and from P, draw PL $\perp$ NQ. Join QP and produce it to meet OX in R; and let $\angle XRP = \theta$. Let PT be the tangent to the curve at P meeting OX in T and let $\angle XTP = \psi$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Now $\quad PL = MN = ON - OM = x + \delta x - x = \delta x$

and $\quad LQ = NQ - NL = NQ - MP = y + \delta y - y = \delta y$

$$\therefore \quad \frac{\delta y}{\delta x} = \tan Q\hat{P}L = \tan X\hat{R}Q = \tan\theta \qquad \ldots(1)$$



Now let Q move along the curve towards P, so that point Q approaches nearer and nearer to the point P and ultimately coincides with it. In the limiting position chord PQ becomes the tangent at P, *i.e.*, PT and it makes an angle ψ with the positive direction of x-axis. Therefore, limiting value of θ is ψ.

Also in the limiting position when $Q \to P$, $\delta x \to 0$ and $\delta y \to 0$

and $\quad \underset{\delta x \to 0}{\text{Lt}} \dfrac{\delta y}{\delta x} = \dfrac{dy}{dx}$.

$$\therefore \quad \tan\psi = \underset{Q \to P}{\text{Lt}} \tan\theta = \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \frac{dy}{dx} \qquad [\therefore \text{ of } (1)]$$

Thus $\dfrac{dy}{dx}$ at any point (x, y) of a curve $y = f(x)$ is the tangent of the angle which the tangent line at P makes with the positive direction of x-axis.

$\therefore \quad \dfrac{dy}{dx}$ at any point (x, y) of any curve $y = f(x)$ is the slope (or gradient) of the tangent to the curve at the point (x, y).

Note. 1. If ψ is the angle which the tangent to the curve makes with x-axis, then

$$\frac{dy}{dx} = \tan\psi.$$

2. If the tangent is parallel to x-axis, then $\psi = 0$

$$\therefore \quad \tan\psi = \tan 0 = 0 \qquad \Rightarrow \qquad \frac{dy}{dx} = 0$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

3. If the tangent is parallel to y-axis, then $\psi = 90°$

$\therefore \quad \cot\psi = \cot 90° = 0$

$\therefore \quad \dfrac{1}{\dfrac{dy}{dx}} = 0 \qquad$ or $\qquad \dfrac{dx}{dy} = 0$

Art-3. "Every real valued function, which is derivable at a point of its domain is also continuous at that point. Prove it. Is the converse true ?

Proof : Let f be derivable at $x = c$

$$\therefore \quad f'(c) = \underset{x \to c}{\text{Lt}} \frac{f(x) - f(c)}{x - c} \qquad \ldots(1)$$

where $f'(c)$ is finite.

Now $\quad f(x) - f(c) = \dfrac{f(x) - f(c)}{x - c}(x - c)$

$$\therefore \quad \underset{x \to c}{\text{Lt}} [f(x) - f(c)] = \underset{x \to c}{\text{Lt}} \left[\frac{f(x) - f(c)}{x - c} \cdot (x - c)\right]$$

$$= \underset{x \to c}{\text{Lt}} \frac{f(x) - f(c)}{x - c} \cdot \underset{x \to c}{\text{Lt}} (x - c)$$

$$= f'(c) \times 0 \qquad [\because \text{ of } (1)]$$

$$\therefore \quad \underset{x \to c}{\text{Lt}} [f(x) - f(c)] = 0$$

$$\Rightarrow \quad \underset{x \to c}{\text{Lt}} f(x) - f(c) = 0 \qquad \Rightarrow \quad \underset{x \to c}{\text{Lt}} f(x) = f(c)$$

$\Rightarrow$ f is continuous at $x = c$

Hence the result.

Converse is not necessarily true.

A function which is continuous at a point may or may not be derivable at that point. For example, let $f(x) = |x|$

We will show that at origin, it is continuous but not derivable.

$$\underset{x \to 0-}{\text{Lt}} f(x) = \underset{x \to 0-}{\text{Lt}} |x| = \underset{h \to 0}{\text{Lt}} |0 - h| \qquad [\text{Putting } x = 0 - h,\ h > 0]$$

$$= \underset{h \to 0}{\text{Lt}} |-h| = \underset{h \to 0}{\text{Lt}} h = 0$$

$$\underset{x \to 0+}{\text{Lt}} f(x) = \underset{x \to 0+}{\text{Lt}} |x| = \underset{h \to 0}{\text{Lt}} |0 + h| \qquad [\text{Putting } x = 0 + h,\ h > 0]$$

$$= \underset{h \to 0}{\text{Lt}} |h| = \underset{h \to 0}{\text{Lt}} h = 0$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Also $f(0) = 0$

$$\therefore \quad \underset{x \to 0-}{\text{Lt}} f(x) = \underset{x \to 0+}{\text{Lt}} f(x) = f(0)$$

$\therefore$ $f(x)$ is continuous at $x = 0$.

$$\text{L.H.D.} = \underset{x \to 0-}{\text{Lt}} \frac{f(x) - f(0)}{x - 0} = \underset{x \to 0-}{\text{Lt}} \frac{|x| - 0}{x - 0}$$

$$= \underset{x \to 0-}{\text{Lt}} \frac{|x|}{x} = \underset{h \to 0}{\text{Lt}} \frac{|0 - h|}{-h} \qquad [\text{ Putting } x = 0 - h,\ h > 0]$$

$$= \underset{h \to 0}{\text{Lt}} \frac{|-h|}{-h} = \underset{h \to 0}{\text{Lt}} \frac{h}{-h} = -1$$

$$\text{R.H.D.} = \underset{x \to 0+}{\text{Lt}} \frac{f(x) - f(0)}{x - 0} = \underset{x \to 0+}{\text{Lt}} \frac{|x| - 0}{x - 0}$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{|x|}{x} = \underset{h \to 0}{\text{Lt}} \frac{|0 + h|}{0 + h} \qquad [\text{Putting } x = 0 + h,\ h > 0]$$

$$= \underset{h \to 0}{\text{Lt}} \frac{|h|}{h} = \underset{h \to 0}{\text{Lt}} \frac{h}{h} = 1$$

$\therefore$ L.H.D. $\neq$ R.H.D.

$\therefore$ $f(x)$ is not derivable at $x = 0$

$\therefore$ $f(x)$ is continuous at $x = 0$ but not derivable at $x = 0$

Art-4. Differentiability and Differentials

A function $y = f(x)$ is said to be differentiable at a point x of the domain $[a, b]$, if it is finite and determinate in any neighbourhood of that point and if the increment δy in the function corresponding to any arbitrarily assigned increment δx in x is expressed in the form

$$\delta y = f(x + \delta x) - f(x) = \text{A}\ \delta x + \varepsilon\ \delta x \qquad \ldots(1)$$

where A is independent of δx and ε is a function of δx which approaches zero as $\delta x \to 0$.

In (1), A δx is called the differential of y at (x, y) and is denoted by dy

$$\therefore \quad dy = \text{A}\ \delta x \qquad \ldots(2)$$

$$\therefore \quad \delta y = dy + \varepsilon\ \delta x \qquad \ldots(3)$$

From (1), $\dfrac{\delta y}{\delta x} = \text{A} + \varepsilon$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$\therefore \quad \underset{\delta x \to 0}{\text{Lt}} \dfrac{\delta y}{\delta x} = \text{A}$, as $\varepsilon \to 0$ as $\delta x \to 0$

$\therefore \quad \dfrac{dy}{dx} = f'(x)$

= Differential coefficient of y at (x, y)

$\therefore$ if $f(x)$ is differentiable then $f'(x)$ exists and has a finite value A.

Art-5. Infinitesimals

An infinitesimal is a variable which tends to zero and is not a number however small.

$x - a$ is infinitesimal as $x \to a$.

Two infinitesimals $f(x)$ and $g(x)$ are of the same order

if $\dfrac{f(x)}{g(x)} \to \text{A}$, a finite limit

If A = 1, then $f(x)$ and $g(x)$ are said to be equivalent infinitesimals.

Note : We know that

$$\delta y = \text{A}\,\delta x + \varepsilon\,\delta x \quad \text{where } \varepsilon \text{ to as } \delta x \to 0 \qquad \ldots(1)$$

$$\therefore \quad \frac{\delta y}{\delta x} = \text{A} + \varepsilon$$

$$\therefore \quad \frac{\delta y}{\delta x} = f'(x) + \varepsilon \qquad \ldots(2)$$

Also $\quad dy = A\,\delta x \qquad \ldots(3)$

From (1) and (3),

$$\frac{\delta y}{dy} = 1 + \frac{\varepsilon}{f'(x)}$$

$$\therefore \quad \frac{\delta y}{\delta x} \to 1 \text{ as } \varepsilon \to 0 \text{ when } \delta x \to 0$$

$\therefore$ when δx is infinitesimal, δy and dy are two equivalent infinitesimals.

Art-6. The necessary and sufficient condition for a function $f(x)$ to be differentiable at a given point x is that the function $f(x)$ posseses a finite derivative at the point.

Proof. Condition is necessary

Let $f(x)$ be differentiable at a given point x.

$$\therefore \quad f(x + \delta x) - f(x) = \text{A}\,\delta x + \varepsilon\,\delta x \qquad \ldots(1)$$

where $\varepsilon \to 0$ as $\delta x \to 0$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\therefore \quad \frac{f(x+\delta x)-f(x)}{x} = A + \varepsilon$$

$$\therefore \quad \underset{\delta x \to 0}{Lt} \frac{f(x+\delta x)-f(x)}{x} = A, \text{ as } \varepsilon \to 0 \text{ when } \delta x \to 0$$

$$\therefore \quad f'(x) = A$$

$\therefore$ $f(x)$ possesses a finite derivative A at x.

Condition is sufficient

Let $f(x)$ possess a finite derivative $f'(x)$ at x.

$$\therefore \quad \underset{\delta x \to 0}{Lt} \frac{f(x+\delta x)-f(x)}{\delta x} = f'(x), \text{ which is finite.}$$

Put $$\frac{f(x+\delta x)-f(x)}{\delta x} - f'(x) = \varepsilon$$

so that $\varepsilon \to 0$ as $\delta x \to 0$

$$\therefore \quad f(x+\delta x) - f(x) = \delta x\, f'(x) + \varepsilon\, \delta x$$

$\therefore$ $f(x)$ is differentiable at x.

Art-7. Darboux's Theorem

Statement If $f(x)$ is derivable for every point x in a closed interval $[a, b]$ and $f'(a), f'(b)$ are of opposite signs, then there exists at least one point x_0 of the interval $[a, b]$ such that $f'(x) = 0$.

Proof. Without any loss of generality, we assume that $f'(a)$ is positive and $f'(b)$ is negative.

Put $$F(x, \delta x) = \frac{f(x+\delta x)-f(x)}{\delta x} \qquad \ldots(1)$$

Since $f(x)$ is derivable in $[a, b]$

$\therefore$ $f(x)$ is continuous in $[a, b]$ and consequently $F(x, \delta x)$ is continuous in $[a, b]$ for $\delta x \neq 0$.

Since $f'(a)$ and $f'(b)$ are of opposite signs and since $F(x, \delta x)$ is also a continuous function in the interval $a \le x \le b$ for $\delta x \neq 0$

$\therefore$ $F(x, \delta x)$ vanishes at some point x_0 intermediate between a and b.

$$\therefore \quad F(x_0, \delta x) = 0 \Rightarrow \frac{f(x_0+\delta x)-f(x)}{\delta x} = 0 \qquad [\because \text{ of } (1)]$$

$$\therefore \quad f'(x_0) = 0$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Cor. If $f(x)$ is derivable in $[a, b]$ and $f'(a) \neq f'(b)$, then there exists at least one point x_0 of the interval $[a, b]$ such that

$f'(x) = A$ where A is any number lying between $f'(a)$ and $f'(b)$.

Proof. Consider $\phi(x) = f(x) - Ax$ ...(1)

$\because$ $f(x)$ is derivable in $[a, b]$

$\therefore$ $\phi(x)$ is also derivable in $[a, b]$ *i.e.* it is a continuous function in $[a, b]$ and has a derivative at every point of $[a, b]$.

Now $\phi'(x) = f'(x) - A$

$\therefore$ $\phi'(a) = f'(a) - A$ and $\phi'(b) = f'(b) - A$ are of opposite signs.

$\therefore$ by Darboux's Theorem, there exists at least one point x_0 of $[a, b]$ such that

$\phi'(x_0) = 0$

$\therefore$ $f'(x_0) - A = 0$, or $f'(x_0) = A$

Cor. If $f(x)$ is derivable in $[a, b]$, then $f'(x)$ cannot pass from a value A to value B without taking all intermediate values.

Proof. By the result of above cor., we can find x_1, x_2 such that

$f'(x_1) = A$, $f'(x_2) = B$ and $A < B$.

Then $f'(x) = 0$ where $x_1 < x_0 < x_2$.

Consider the case when $f'(x)$ does not change sign in passing from A to B. Let C be any value between A and B.

$\therefore$ there exists some point x^* between x_1 and x_2 at which $\phi'(x^*] = 0$

i.e. $\phi'(x^*) = f'(x^*) - C = 0$

$\therefore$ $f'(x^*) = C$

Hence the result.

ILLUSTRATIVE EXAMPLES

Example 1. Let f be differentiable at a, then evaluate $\underset{x \to a}{\text{Lt}} \dfrac{x^2 f(a) - a^2 f(x)}{x - a}$.

Sol. Put $x = a + h$ so that $h \to 0$ as $x \to a$

$$\therefore \quad \underset{x \to a}{\text{Lt}} \frac{x^2 f(a) - a^2 f(x)}{x - a} = \underset{h \to 0}{\text{Lt}} \frac{(a+h)^2 f(a) - a^2 f(a+h)}{a + h - a}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$= \underset{h\to 0}{\text{Lt}} \frac{(a^2+2ah+h^2)f(a)-a^2 f(a+h)}{h}$$

$$= \underset{h\to 0}{\text{Lt}} \frac{a^2 f(a)+2ahf(a)+h^2 f(a)-a^2 f(a+h)}{h}$$

$$= \underset{h\to 0}{\text{Lt}} \frac{h\{2af(a)+hf(a)\}-a^2\{f(a+h)-f(a)\}}{h}$$

$$= \underset{h\to 0}{\text{Lt}} \left[2af(a)+hf(a)-a^2\frac{f(a+h)-f(a)}{h}\right]$$

$$= \underset{h\to 0}{\text{Lt}}\, 2af(a) + \underset{h\to 0}{\text{Lt}}\, hf(a) - a^2 \underset{h\to 0}{\text{Lt}} \frac{f(a+h)-f(a)}{h}$$

$$= 2af(a)+0f(a)-a^2 f'(a) = 2af(a)-a^2 f'(a).$$

Example 2. (*a*) Prove that $f(x) = |x-3|$ has no derivative at $x = 3$.

(*b*) If $f(x) = \begin{cases} x\sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

then show that f is not differentiable at $x = 0$.

(*c*) Prove that the greatest integer function $[x]$ is not differentiable at $x = 1$.

Sol. (*a*) Here $f(x) = |x-3|$

$$\text{L.H.D.} = \underset{x\to 3-}{\text{Lt}} \frac{f(x)-f(3)}{x-3} = \underset{x\to 3-}{\text{Lt}} \frac{|x-3|-0}{x-3} \qquad [\because f(3) = |3-3| = |0| = 0]$$

$$= \underset{x\to 3-}{\text{Lt}} \frac{|x-3|}{x-3} \qquad [\text{Put } x = 3-h,\ h>0 \text{ so that } h\to 0 \text{ as } x\to 3-]$$

$$= \underset{h\to 0}{\text{Lt}} \frac{|3-h-3|}{3-h-3} = \underset{h\to 0}{\text{Lt}} \frac{|-h|}{-h} = \underset{h\to 0}{\text{Lt}} \frac{h}{-h} = -1$$

$$\text{R.H.D} = \underset{x\to 3+}{\text{Lt}} \frac{f(x)-f(3)}{x-3} = \underset{x\to 3+}{\text{Lt}} \frac{|x-3|-0}{x-3}$$

$$= \underset{x\to 3+}{\text{Lt}} \frac{|x-3|}{x-3} \qquad [\text{Put } x = 3+h,\ h>0 \text{ so that } h\to 0 \text{ as } x\to 3+]$$

$$= \underset{h\to 0}{\text{Lt}} \frac{|3+h-3|}{3+h-3} = \underset{h\to 0}{\text{Lt}} \frac{|h|}{h} = \underset{h\to 0}{\text{Lt}} \frac{h}{h} = 1$$

$\therefore$ L.H.D. $\neq$ R.H.D.

$\therefore$ $f(x) = |x-3|$ is not derivable at $x = 3$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

(*b*) Here $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$$\text{R.H.D.} = \underset{x \to 0+}{\text{Lt}} \frac{f(x) - f(0)}{x - 0} = \underset{x \to 0+}{\text{Lt}} \frac{x \sin \frac{1}{x} - 0}{x} = \underset{x \to 0+}{\text{Lt}} \sin \frac{1}{x},$$

which does not exist.

$\therefore$ f is not differentiable at $x = 0$.

(*c*) Let $f(x) = [x]$

$$\underset{x \to 1-}{\text{Lt}} f(x) = \underset{x \to 1-}{\text{Lt}} [x] \quad [\text{Put } x = 1 - h,\ h > 0 \text{ so that } h \to 0 \text{ as } x \to 1-]$$

$$= \underset{h \to 0}{\text{Lt}} [1 - h] = \underset{h \to 0}{\text{Lt}} (0) = 0$$

$$\underset{x \to 1+}{\text{Lt}} f(x) = \underset{x \to 1+}{\text{Lt}} [x] \quad [\text{Put } x = 1 + h,\ h > 0 \text{ so that } h \to 0 \text{ as } x \to 1+]$$

$$= \underset{h \to 0}{\text{Lt}} [1 + h] = \underset{h \to 0}{\text{Lt}} (1) = 1$$

$\therefore$ $\underset{x \to 1-}{\text{Lt}} f(x) \neq \underset{x \to 1+}{\text{Lt}} f(x)$

$\Rightarrow$ f is not continuous at $x = 1$

$\Rightarrow$ f is not differentiable at $x = 1$.

Example 3. A function f is defined as $f(x) = \begin{cases} 1 + x, & \text{iff } x \leq 2 \\ 5 - x, & \text{iff } x > 2 \end{cases}$

Show that f is not differentiable at $x = 2$.

Sol. Here $f(x) = \begin{cases} 1 + x, & \text{iff } x \leq 2 \\ 5 - x, & \text{iff } x > 2 \end{cases}$

$$\text{L.H.D.} = \underset{x \to 2-}{\text{Lt}} \frac{f(x) - f(2)}{x - 2} = \underset{x \to 2-}{\text{Lt}} \frac{(1 + x) - 3}{x - 2} \quad [\because f(2) = 1 + 2 = 3]$$

$$= \underset{x \to 2-}{\text{Lt}} \frac{x - 2}{x - 2} = 1$$

$$\text{R.H.D.} = \underset{x \to 2+}{\text{Lt}} \frac{f(x) - f(2)}{x - 2} = \underset{x \to 2+}{\text{Lt}} \frac{(5 - x) - 3}{x - 2} = \underset{x \to 2+}{\text{Lt}} \frac{2 - x}{x - 2}$$

$$= \underset{x \to 2+}{\text{Lt}} \frac{-(x + 2)}{x - 2} = -1$$

$\therefore$ L.H.D. $\neq$ R.H.D.

$\therefore$ f is not derivable at $x = 2$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 4. Determine the set of all points where the function $f(x) = \dfrac{x}{1+|x|}$ is differentiable.

Sol. Here $f(x) = \dfrac{x}{1+|x|}$, $D_f = R$

For $x \geq 0$, we have $|x| = x$ $\Rightarrow$ $f(x) = \dfrac{x}{1+x}$

For $x > 0$, we have $|x| = -x$ $\Rightarrow$ $f(x) = \dfrac{x}{1-x}$

$$\therefore \quad f(x) = \begin{cases} \dfrac{x}{1+x}, & x \geq 0 \\ \dfrac{x}{1-x}, & x < 0 \end{cases}$$

Clearly $f(x)$ is differentiable for $x < 0$ and $x > 0$.

$$Lf'(0) = \underset{h \to 0-}{\text{Lt}} \frac{f(0+h) - f(0)}{h} = \underset{h \to 0-}{\text{Lt}} \frac{\dfrac{h}{1+h} - \dfrac{0}{1+0}}{h} = \underset{h \to 0-}{\text{Lt}} \frac{1}{1-h} = \frac{1}{1-0} = 1$$

$$Rf'(0) = \underset{h \to 0+}{\text{Lt}} \frac{f(0+h) - f(0)}{h} = \underset{h \to 0+}{\text{Lt}} \frac{\dfrac{h}{1+h} - \dfrac{0}{1+0}}{h} = \underset{h \to 0+}{\text{Lt}} \frac{1}{1+h} = \frac{1}{1+0} = 1$$

$\therefore$ $Lf'(0) = Rf'(0)$

$\therefore$ $f(x)$ is differentiable at $x = 0$

$\therefore$ $f(x)$ is differentiable for all finite values of x.

Example 5. For what choices of a and b is the function

$$f(x) = \begin{cases} x^2, & x \leq c \\ ax + b, & x > c \end{cases}$$

differentiable at $x = c$?

Sol. Here $f(x) = \begin{cases} x^2, & x \leq c \\ ax + b, & x > c \end{cases}$

Since f is differentiable at $x = c$

$\therefore$ f is continuous at $x = c$

$$\therefore \quad \underset{x \to c-}{\text{Lt}} f(x) = \underset{x \to c+}{\text{Lt}} f(x) = f(c)$$

$$\Rightarrow \quad \underset{x \to c-}{\text{Lt}} x^2 = \underset{x \to c+}{\text{Lt}} (ax + b) = c^2$$

$$\Rightarrow \quad c^2 = ac + b = c^2 \quad \Rightarrow \quad ac + b = c^2 \qquad \ldots(1)$$

Since f is differentiable at $x = c$

$$\therefore \quad \underset{x \to c-}{\text{Lt}} \frac{f(x) - f(c)}{x - c} = \underset{x \to c+}{\text{Lt}} \frac{f(x) - f(c)}{x - c}$$

$$\Rightarrow \quad \underset{x \to c-}{\text{Lt}} \frac{x^2 - c^2}{x - c} = \underset{x \to c+}{\text{Lt}} \frac{(ax + b) - c^2}{x - c}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\Rightarrow \quad \underset{h \to 0}{\text{Lt}} \frac{(c-h)^2 - c^2}{c-h-c} = \underset{h \to 0}{\text{Lt}} \frac{\{a(c+h)+b\} - c^2}{c+h-c}$$

$$\Rightarrow \quad \underset{h \to 0}{\text{Lt}} \frac{h^2 - 2ch}{-h} = \underset{h \to 0}{\text{Lt}} \frac{ac+b+ah-c^2}{h}$$

$$\Rightarrow \quad \underset{h \to 0}{\text{Lt}} \frac{h^2 - 2ch}{-h} = \underset{h \to 0}{\text{Lt}} \frac{ah}{h} \qquad [\because \text{ of } (1)]$$

$$\Rightarrow \quad \underset{h \to 0}{\text{Lt}} (-h+2c) = \underset{h \to 0}{\text{Lt}}\, a$$

$\Rightarrow \quad -0+2c = a \qquad \Rightarrow \quad a = 2c$

$\therefore \quad$ from (1), $2c \,.\, c + b = c^2 \qquad \Rightarrow \quad b = -c^2$

Hence $a = 2c$, $b = -c^2$

EXERCISE 2 (*a*)

1. If f is differentiable at $x = a$, then prove that

$$\underset{x \to a}{\text{Lt}} \frac{x f(a) - a f(x)}{x - a} = f(a) - a f'(a)$$

2. Examine the derivability of the following functions :

   (*i*) $|x|$ at $x = 0$ (*ii*) $|x^2|$ at $x = 0$

3. Examine the derivability of the following functions :

   (*i*) $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

   at $x = 0$

   (*ii*) $f(x) = \begin{cases} 3 - 2x, & x < 2 \\ 3x - 7, & x \geq 2 \end{cases}$

   at $x = 2$

   (*iii*) $f(x) = \begin{cases} x - 1, & x < 2 \\ 2x - 3, & x \geq 2 \end{cases}$

   at $x = 2$

4. Investigate the following function from the point of view of its differentiability. Does the differential coefficient of the function exist at $x = 0$ and $x = 1$?

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ x^3 - x + 1, & x > 1 \end{cases}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

5. (*a*) The function f defined by $f(x) = \begin{cases} x^2 + 3x + a, & \text{if } x \le 1 \\ bx + 2, & \text{if } x > 1 \end{cases}$

is given to be derivable for all x. Find a and b.

(*b*) The function f defined by $f(x) = \begin{cases} ax^2 + 1, & x \ge 1 \\ x + a, & x < 1 \end{cases}$

is given to be derivable at $x = 1$. Find a.

(*c*) Find the values of a and b so that $f(x) = \begin{cases} x^3, & x \le 2 \\ ax^2 + bx, & x > 2 \end{cases}$

becomes differentiable on the whole line.

6. If $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbf{R}$ and $f(5) = 2$, $f'(0) = 3$. Find $f'(5)$.

7. If $f(a) = 2$, $f'(a) = 1$, $g(a) = -1$, $g'(a) = 2$, then find the value of

$$\underset{x \to a}{\text{Lt}} \frac{g(x)f(a) - g(a)f(x)}{x - a}$$

8. If $f(9) = 9$ and $f'(9) = 4$, then evaluate $\underset{x \to 9}{\text{Lt}} \dfrac{\sqrt{f(x)} - 3}{\sqrt{x} - 3}$.

9. Let $f(x) = \begin{cases} 2x + 3 & \text{for } x \le 1 \\ ax^2 + bx & \text{for } x > 1 \end{cases}$

If $f(x)$ is everywhere differentiable, then prove that $f'(2) = -4$.

10. Let $f(x) = \begin{cases} x e^{-\left(\frac{1}{|x|} + \frac{1}{x}\right)} & , x \ne 0 \\ 0 & , x = 0 \end{cases}$

Test whether $f(x)$ is differentiable at $x = 0$.

11. Determine the values of x for which the following function fails to be continuous or differentiable :

$$f(x) = \begin{cases} 1 - x & , x < 1 \\ (1-x)(2-x) & , 1 \le x \le 2 \\ 3 - x & , x > 2 \end{cases}$$

Justify your answer.

12. Examine the right-handed and left-handed derivatives at the origin for the function

$$f(x) = \begin{cases} \dfrac{x}{1 + e^{\frac{1}{x}}} & , x \ne 0 \\ 0 & , x = 0 \end{cases}$$

Is function derivable at $x = 0$?

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

13. Calculate the right-handed and left-handed derivative at origin for the function

$$f(x) = \begin{cases} x \tan^{-1} \dfrac{1}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

is $f(x)$ derivable at $x = 0$.

14. The function $f(x) = |x| + |x - 1|$ is non-derivable at $x = 0$ and $x = 1$ although it is continuous at these points.

ANSWERS

2. (*i*) Not differentiate (*ii*) Derivable

3. (*i*) Derivable (*ii*) Not derivative (*iii*) Not derivative

4. Not differentiable at $x = 0$; differentiable at $x = 1$

5. (*a*) $a = 3,\ b = 5$ (*b*) $a = \dfrac{1}{2}$ (*c*) $a = 4, b = -4$

6. 6 **7.** 5 **8.** 4

10. Not differentiable at $x = 0$

11. Continuous and differentiable at $x = 1$; Not continuous and differentiable at $x = 2$

12. Not derivable at $x = 0$ **13.** Not differentiable at $x = 0$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

3

SUCCESSIVE DIFFERENTIATION

Art-1. Definition and Notation

If y is a function of x, which possesses differential coefficient, then we know that the differential coefficient is denoted by $\frac{dy}{dx}$ and is called the first differential coefficient of y *w.r.t.* x. In general this differential coefficient $\frac{dy}{dx}$ is again a function of x and can be differentiated again. The differential coefficient thus obtained is called the second differential coefficient of y *w.r.t.* x, and is denoted by $\frac{d^2y}{dx^2}$ (read as 'd to y by dx square').

Similarly the differential coefficient of $\frac{d^2y}{dx^2}$ is called the third differential coefficient of y *w.r.t.* x and written as $\frac{d^3y}{dx^3}$ (read as 'd three y by dx cube'), and so on. In general nth differential coefficient is denoted by $\frac{d^ny}{dx^n}$. The process of repeated differentiation is called successive differentiation. Other notations for successive differential coefficients of y are

$$y_1, y_2, y_3, \ldots, y_n \quad \text{or} \quad y', y'', y''', \ldots, y^{(n)} \quad \text{or} \quad Dy, D^2y, D^3y, \ldots, D^ny.$$

Similarly successive differential coefficients of function $f(x)$ are denoted by

$$f'(x), f''(x), f'''(x), \ldots, f^{(n)}(x).$$

Note. The symbol $\frac{d^n}{dx^n}$ written before a function of x indicates that the function is to be differentiated n times in succession.

ILLUSTRATIVE EXAMPLES

Example 1. Find the fourth derivative of $\log\sqrt{3x+4}$.

Sol. Let $y = \log\sqrt{3x+4} = \log(3x+4)^{\frac{1}{2}} = \frac{1}{2}\log(3x+4)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS MoE 2025 Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}\left[\frac{3}{3x+4}\right] = \frac{3}{2}(3x+4)^{-1}$$

Again differentiating w.r.t. x,

$$\frac{d^2y}{dx^2} = \frac{3}{2}(-1)(3x+4)^{-2} \times 3 = -\frac{9}{2}(3x+4)^{-2}$$

Differentiating again w.r.t. x,

$$\frac{d^3y}{dx^3} = -\frac{9}{2}(-2)(3x+4)^{-3}(3) = 27(3x+4)^{-3}$$

Differentiating again w.r.t. x,

$$\frac{d^4y}{dx^4} = 27(-3)(3x+4)^{-4}(3) = -243(3x+4)^{-4} = -\frac{243}{(3x+4)^4}.$$

Example 2. If $f(x) = \tan x$, show that $f^{v}(0) = 16$.

Sol. $f(x) = \tan x$ ...(1)

$\therefore \quad f'(x) = \sec^2 x = 1 + \tan^2 x$ ...(2)

$\therefore \quad f''(x) = 0 + 2\tan x \sec^2 x = 2\tan x(1 + \tan^2 x) = 2\tan x + 2\tan^3 x$

$f''(x) = 2f(x) + 2(f(x))^3$ ...(3)

$\therefore \quad f'''(x) = 2f'(x) + 6(f(x))^2 . f'(x)$ ...(4)

$f^{iv}(x) = 2f''(x) + 12f(x)(f'(x))^2 + 6(f(x))^2 f''(x)$ ...(5)

$\therefore \quad f^{v}(x) = 2f'''(x) + 12(f'(x))^3 + 12 \times 2f(x)f'(x)f''(x)$

$+ 12f(x)f'(x)f''(x) + 6(f(x))^2 f'''(x)$ ...(6)

Putting $x = 0$ in equations (1) to (6), we get,

$f(0) = \tan 0 = 0$

$f'(0) = 1 + \tan^2 0 = 1 + 0 = 1$

$f''(0) = 2f(0) + 2(f(0))^3 = 2 \times 0 + 2 \times 0 = 0$

$f'''(0) = 2f'(0) + 6(f(0))^2 f'(0) = 2 \times 1 + 6 \times 0 \times 1 = 2$

$f^{iv}(0) = 2f''(0) + 12f(0)(f'(0))^2 + 6(f(0))^2 f''(0)$

$= 2 \times 0 + 12 \times 0 \times 1 + 6 \times 0 \times 0 = 0$

$f^{v}(0) = 2f'''(0) + 12(f'(0))^3 + 24f(0)f'(0)f''(0)$

$+ 12f(0)f'(0)f''(0) + 6(f(0))^2 f'''(0)$

$= 2 \times 2 + 12 \times (1)^3 + 0 + 0 + 0 = 4 + 12 = 16.$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 3. If $y = e^x \sin x$, prove that $\dfrac{d^2y}{dx^2} = 2e^x \cos x$.

Sol. $$y = e^x \sin x$$

$$\therefore \quad \frac{dy}{dx} = e^x \frac{d}{dx}(\sin x) + \sin x \cdot \frac{d}{dx}(e^x)$$

$$= e^x \cos x + \sin x \cdot e^x = e^x (\cos x + \sin x)$$

$$\therefore \quad \frac{d^2y}{dx^2} = e^x \frac{d}{dx}(\cos x + \sin x) + (\cos x + \sin x)\frac{d}{dx}(e^x)$$

$$= e^x(-\sin x + \cos x) + (\cos x + \sin x) \cdot e^x$$

$$= e^x(-\sin x + \cos x + \cos x + \sin x) = e^x (2 \cos x)$$

$$\therefore \quad \frac{d^2y}{dx^2} = 2e^x \cos x.$$

Example 4. If $y = \sin(m \sin^{-1} x)$, prove that $(1-x^2)\, y_2 - x\, y_1 + m^2 y = 0$.

Sol. Here $y = \sin(m \sin^{-1} x)$

$$\therefore \quad y_1 = \cos(m \sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}} \quad \Rightarrow \quad y_1 \sqrt{1-x^2} = m \cos(m \sin^{-1} x)$$

$$\Rightarrow (1-x^2) \cdot y_1^2 = m^2 \cos^2(m \sin^{-1} x) \quad \Rightarrow \quad (1-x^2) \cdot y_1^2 = m^2 [1 - \sin^2(m \sin x)]$$

$$\Rightarrow (1-x^2) \cdot y_1^2 = m^2 (1-y^2) \qquad [\because \text{ of } (1)]$$

Differentiating both sides *w.r.t.* x,

$$(1-x^2)\, 2 y_1 y_2 + y_1^2 (-2x) = m^2 (-2 y y_1)$$

Dividing both sides by $2y_1$, we get,

$$(1-x^2)\, y_2 - x\, y_1 = -m^2 y \quad \text{or} \quad (1-x^2)\, y_2 - x\, y_1 + m^2 y = 0$$

Example 5. If $y = \dfrac{ax+b}{cx+d}$, prove that $2 y_1 y_3 = 3 y_2^2$.

Sol. Here $$y = \frac{ax+b}{cx+d}$$

$$\therefore \quad y_1 = \frac{(cx+d).a - (ax+b).c}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2}$$

$$\therefore \quad y_1 = (ad-bc)(cx+d)^{-2} \qquad \ldots(1)$$

$$y_2 = -2c(ad-bc)(cx+d)^{-3} \qquad \ldots(2)$$

$$y_3 = 6c^2(ad-bc)(cx+d)^{-4} \qquad \ldots(3)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

L.H.S. $= 2\,y_1\,y_3 = 2 \times [(a\,d - b\,c)\,(c\,x + d)^{-2}] \times [6\,c^2\,(a\,d - b\,c)\,(c\,x + d)^{-4}]$

$= 12\,c^2\,(a\,d - b\,c)^2\,(c\,x + d)^{-6} = 3 \times [-2\,c\,(a\,d - b\,c)\,(c\,x + d)^{-3}]^2$

$= 3\,y_2^2$ [$\because$ of (2)]

= R.H.S.

Example 6. If $p^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$, prove that $p + \dfrac{d^2 p}{d\theta^2} = \dfrac{a^2 b^2}{p^3}$.

Sol. Here $p^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$

$$\Rightarrow \quad p^2 = a^2 \left(\frac{1 + \cos 2\theta}{2}\right) + b^2 \left(\frac{1 - \cos 2\theta}{2}\right)$$

$$\Rightarrow \quad 2\,p^2 = a^2\,(1 + \cos 2\theta) + b^2\,(1 - \cos 2\theta)$$

$$\Rightarrow \quad 2\,p^2 - (a^2 + b^2) = (a^2 - b^2)\cos 2\theta \qquad \ldots(1)$$

Differentiating *w.r.t.* θ, we get,

$$4\,p\,\frac{dp}{d\theta} = -2\,(a^2 - b^2)\sin 2\theta$$

or $$-2\,p\,\frac{dp}{d\theta} = (a^2 - b^2)\sin 2\theta \qquad \ldots(2)$$

Squaring (1), (2) and adding, we get,

$$4\,p^4 + (a^2 + b^2)^2 - 4\,p^2\,(a^2 + b^2) + 4\,p^2 \left(\frac{dp}{d\theta}\right)^2 = (a^2 - b^2)^2$$

or $$4\,p^4 - 4\,p^2\,(a^2 + b^2) + 4\,p^2 \left(\frac{dp}{d\theta}\right)^2 + (a^2 + b^2)^2 - (a^2 - b^2)^2 = 0$$

or $$4\,p^4 - 4\,p^2\,(a^2 + b^2) + 4\,p^2 \left(\frac{dp}{d\theta}\right)^2 + 4\,a^2\,b^2 = 0$$

Dividing both sides $4\,p^2$, we get,

$$p^2 - (a^2 + b^2) + \left(\frac{dp}{d\theta}\right)^2 + \frac{a^2 b^2}{p^2} = 0$$

Differentiating again *w.r.t.* θ, we get,

$$2\,p\,\frac{dp}{d\theta} + 2\,\frac{dp}{d\theta}\,\frac{d^2 p}{d\theta^2} - \frac{2\,a^2 b^2}{p^3}\,\frac{dp}{d\theta} = 0$$

Dividing by $2\,\dfrac{dp}{d\theta}$, we get,

$$p + \frac{d^2 p}{d\theta^2} = \frac{a^2 b^2}{p^3}.$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 7. If $x = a(\cot t + t \sin t)$, $y = a(\sin t - t \cos t)$, compute $\dfrac{d^2y}{dx^2}$ at $t = \dfrac{\pi}{4}$.

Sol.

$$x = a(\cos t + t \sin t)$$

$$\therefore \quad \frac{dx}{dt} = a(-\sin t + t \cos t + \sin t)$$

$$= a(t \cos t)$$

$$\therefore \quad \frac{dx}{dt} = a\, t \cos t$$

$$y = a(\sin t - t \cos t)$$

$$\therefore \quad \frac{dy}{dt} = a(\cos t + t \sin t - \cos t)$$

$$= a(t \sin t)$$

$$\therefore \quad \frac{dy}{dt} = a\, t \sin t$$

Now $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\, t \sin t}{a\, t \cos t} = \tan t$$

$$\therefore \quad \frac{d^2y}{dx^2} = \sec^2 t \,.\, \frac{dt}{dx} = \sec^2 t \,.\, \frac{1}{a\, t \cos t} = \frac{1}{a\, t \cos^3 t}$$

At $t = \dfrac{\pi}{4}$,

$$\frac{d^2y}{dx^2} = \frac{1}{a\frac{\pi}{4}\cos^3\frac{\pi}{4}} = \frac{1}{a \times \frac{\pi}{4} \times \frac{1}{2\sqrt{2}}} = \frac{8\sqrt{2}}{a\,\pi}$$

Example 8. If $x = \sin\theta$, $y = \sin m\theta$, prove that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2 y = 0.$$

Sol. $x = \sin\theta$, $y = \sin m\theta$

$$\therefore \quad \frac{dx}{d\theta} = \cos\theta, \quad \frac{dy}{d\theta} = m \cos m\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{m \cos m\theta}{\cos\theta} = \frac{m\sqrt{1 - \sin^2 m\theta}}{\sqrt{1 - \sin^2\theta}}$$

$$\therefore \quad y_1 = \frac{m\sqrt{1 - y^2}}{\sqrt{1 - x^2}} \quad \Rightarrow \quad \sqrt{1 - x^2}\,.\, y_1 = m\sqrt{1 - y^2}$$

$$\Rightarrow \quad (1 - x^2)\, y_1^2 = m^2 (1 - y^2).$$

Differentiating both sides *w.r.t.* x, we get,

$$(1 - x^2)\,.\, 2\, y_1 y_2 + y_1^2(-2x) = m^2(-2\, y\, y_1)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Dividing both sides by $2y_1$, we get,

$$(1-x^2)\,y_2 - x\,y_1 = -m^2 y \quad \text{or} \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2 y = 0.$$

Example 9. If $\sqrt{x}+\sqrt{y}=\sqrt{a}$, find the value of $\dfrac{d^2y}{dx^2}$ at $x=a$.

Sol. Here $\sqrt{x}+\sqrt{y}=\sqrt{a}$ ...(1)

Differentiating *w.r.t.* x,

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0 \quad \Rightarrow \quad \frac{1}{2\sqrt{y}}\frac{dy}{dx}=-\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}} \qquad ...(2)$$

Again differentiating *w.r.t.* x, we get,

$$\frac{d^2y}{dx^2}=-\left\{\frac{\sqrt{x}\dfrac{d}{dx}(\sqrt{y})-\sqrt{y}\dfrac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2}\right\}=-\left\{\frac{\sqrt{x}\cdot\dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx}-\sqrt{y}\cdot\dfrac{1}{2\sqrt{x}}}{x}\right\}$$

$$=-\frac{1}{x}\left[\frac{\sqrt{x}}{2\sqrt{y}}\left(-\frac{\sqrt{y}}{\sqrt{x}}\right)-\frac{\sqrt{y}}{2\sqrt{x}}\right] \qquad [\because \text{ of } (2)]$$

$$=-\frac{1}{x}\left[-\frac{1}{2}-\frac{\sqrt{y}}{2\sqrt{x}}\right]=\frac{1}{2x}\left[1+\frac{\sqrt{y}}{\sqrt{x}}\right]=\frac{1}{2x}\left[\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}}\right]=\frac{1}{2x}\left[\frac{\sqrt{a}}{\sqrt{x}}\right]$$

$[\because \text{ of } (1)]$

$$\therefore \quad \frac{d^2y}{dx^2}=\frac{\sqrt{a}}{2x^{\frac{3}{2}}}$$

At $x=a$, $\dfrac{d^2y}{dx^2}=\dfrac{\sqrt{a}}{2a^{\frac{3}{2}}}=\dfrac{1}{2a}$.

Example 10. Find a, b, c so that the function $f(x)=\begin{cases} x^3+3x+5, & x\le 1 \\ ax^2+bx+c, & x>1 \end{cases}$

is twice differentiable at $x=1$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Sol. Here $f(x) = \begin{cases} x^3 + 3x + 5, & x \le 1 \\ ax^2 + bx + c, & x > 1 \end{cases}$

$$\therefore \quad f'(x) = \begin{cases} 3x^2 + 3, & x \le 1 \\ 2ax + b, & x > 1 \end{cases}$$

$$f''(x) = \begin{cases} 6x, & x \le 1 \\ 2a, & x > 1 \end{cases}$$

$\because$ $f''(1)$ exists

$\therefore$ $f'(1)$ exists and $f(x)$ is continuous at $x = 1$.

As $f''(1)$ exists, $\therefore$ $f''(1-) = f''(1+)$

$\Rightarrow$ $6.1 = 2a$ $\Rightarrow$ $2a = 6$ $\Rightarrow$ $a = 3$

As $f'(1)$ exists, $\therefore$ $f'(1-) = f'(1+)$

$\therefore$ $3.1^2 + 3 = 2a.1 + b$ $\Rightarrow$ $3 + 3 = 6 + b$ $\Rightarrow$ $b = 0$

$\because$ $f(x)$ is continuous at $x = 1$

$\therefore$ $\underset{x \to 1-}{\text{Lt}} f(x) = f(1) = \underset{x \to 1+}{\text{Lt}} f(x)$

$\therefore$ $1 + 3 + 5 = a + b + c$ $\Rightarrow$ $9 = 3 + 0 + c$ $\Rightarrow$ $c = 6$

$\therefore$ we have $a = 3$, $b = 0$, $c = 6$.

Example 11. If f is defined by

$$f(x) = \begin{cases} 0 \text{ if } x \le 0 \\ x^3 \text{ if } x > 0. \end{cases}$$

Prove that f' and f'' both exists for each x, whereas $f'''(x)$ exists at each x except 0.

Sol. Here $f(x) = \begin{cases} 0, & x \le 0 \\ x^3, & x > 0 \end{cases}$

$\therefore$ when $x < 0$, $f(x) = 0$, $f'(x) = 0$, $f''(x) = 0$, $f'''(x) = 0$

and when $x > 0$, $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6$

When $x = 0$, $f(0) = 0$

Now $f'(0-) = \underset{x \to 0-}{\text{Lt}} \dfrac{f(x) - f(0)}{x - 0} = \underset{x \to 0-}{\text{Lt}} \dfrac{0 - 0}{x}$ $[\because f(x) = 0$ when $x < 0]$

$= 0$

$$f'(0+) = \underset{x \to 0+}{\text{Lt}} \frac{f(x) - f(0)}{x - 0} = \underset{x \to 0+}{\text{Lt}} \frac{x^3 - 0}{x - 0}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$= \underset{x \to 0+}{\text{Lt}} \frac{x^3}{x} \qquad \text{(Putting } x = 0 + h, h > 0)$$

$$= \underset{h \to 0+}{\text{Lt}} \frac{(0+h)^3}{0+h} = \underset{h \to 0+}{\text{Lt}} \frac{h^3}{h} = \underset{h \to 0+}{\text{Lt}} h^2 = 0$$

$\therefore \quad f'(0-) = f'(0+) = 0$

$\therefore \quad f'(0)$ exists and is equal to 0.

$$\therefore \quad f'(x) = \begin{cases} 0, & x < 0 \\ 0, & x = 0 \\ 3x^2, & x > 0 \end{cases}$$

$\therefore \quad f'(x)$ exists for each value of x.

$$\text{Again } f''(0-) = \underset{x \to 0-}{\text{Lt}} \frac{f'(x) - f'(0)}{x - 0} = \underset{x \to 0-}{\text{Lt}} \frac{0-0}{x}$$

$[\because f'(x) = 0 \text{ when } x < 0]$

$$= 0$$

$$f''(0+) = \underset{x \to 0+}{\text{Lt}} \frac{f'(x) - f'(0)}{x - 0} = \underset{x \to 0+}{\text{Lt}} \frac{3x^2 - 0}{x - 0} = \underset{x \to 0+}{\text{Lt}} (3x)$$

[Putting $x = 0 + h$, $h > 0$]

$$= \underset{h \to 0}{\text{Lt}} \{3(0+h)\}$$

$$= 3(0+0) = 0$$

$\therefore \quad f''(0-) = f''(0+) = 0$

$\therefore \quad f''(0)$ exists and is equal to 0.

$$\therefore \quad f''(x) = \begin{cases} 0, & x < 0 \\ 0, & x = 0 \\ 6x, & x > 0 \end{cases}$$

$\therefore \quad f''(x)$ exists for each value of x.

$$\text{Again } f'''(0-) = \underset{x \to 0-}{\text{Lt}} \frac{f''(x) - f''(0)}{x - 0} = \underset{x \to 0-}{\text{Lt}} \frac{0-0}{x} = 0$$

$$f'''(0+) = \underset{x \to 0+}{\text{Lt}} \frac{f''(x) - f''(0)}{x - 0} = \underset{x \to 0+}{\text{Lt}} \frac{6x - 0}{x} = \underset{x \to 0+}{\text{Lt}} (6) = 6$$

$\therefore \quad f'''(0-1) \neq f'''(0+)$

$\therefore \quad f'''(0)$ does not exist

$\therefore \quad f'''(x)$ exists at each x except 0.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

EXERCISE 3 (a)

1. (*a*) Find the derivatives of all orders for the function defined by :

(*i*) $f(x) = 5x^4 - 3x^2 + \sqrt{2}\,x + \pi \quad \forall\, x \in \mathbf{R}$

(*ii*) $f(x) = x^4 + x^3 + 3x^2 + 7x - 1$

(*b*) Find the derivatives of 2nd order of

(*i*) $f(x) = e^{x \sin x}$ (*ii*) $f(x) = x^2 \log|\cos x|$

(*c*) Find the third derivative of

(*i*) $f(x) = \dfrac{x}{\sqrt{1-x^2}}$ (*ii*) $f(x) = x^2 e^x$

(*iii*) $f(x) = \log(\sin x)$

(*d*) Find the fourth derivative of

(*i*) $x^3 \log x$ (*ii*) $\log(2x+3)^2$

2. (*i*) If $y = a e^{mx} + b e^{-mx}$, then prove that $y_2 - m^2 y = 0$.

(*ii*) If $y = a \cos mx + b \sin mx$, then prove that $y_2 + m^2 y = 0$.

(*iii*) If $y = e^{ax} \sin bx$, prove that $y_2 - 2a y_1 + (a^2 + b^2) y = 0$.

3. (*i*) If $f(x) = a \sin(\log x)$, prove that $x^2 f''(x) + x f'(x) + f(x) = 0$.

(*ii*) If $y = (\log x)^2$, prove that $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 2$.

(*iii*) If $y = \log\left(x + \sqrt{x^2+1}\right)$, prove that $(x^2+1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 0$.

(*iv*) If $y = a\cos(\log x) + b\sin(\log x)$, show that $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$.

4. (*i*) If $y = \sin(\sin x)$, prove that $\dfrac{d^2y}{dx^2} + \tan x . \dfrac{dy}{dx} + y\cos^2 x = 0$.

(*ii*) If $y = (a + bt)e^{nt}$, prove that $\dfrac{d^2y}{dx^2} - 2n\dfrac{dy}{dt} + n^2 y = 0$

(*iii*) If $y = \sin(\log x)$, show that $x^2 y_2 + x y_1 + y = 0$

(*iv*) If $y = x \log\left[(px)^{-1} + p^{-1}\right]$, prove that $x(x+1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y + 1 = 0$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

5. (*i*) If $y = \tan^{-1} x$, show that $(1 + x^2)\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} = 0$

(*ii*) If $y = (\tan^{-1} x)^2$, prove that $(x^2 + 1)^2 y_2 + 2x(x^2 + 1) y_1 = 2$

(*iii*) If $y = \cos^{-1} x$, show that $y_2(1 - x^2) - x y_1 = 0$

6. If $y = \dfrac{\sin^{-1} x}{\sqrt{1 - x^2}}$, show that

(*i*) $(1 - x^2) y_1 - x y = 1$ (*ii*) $(1 - x^2) y_2 - 3x y_1 - y = 0$

7. (*i*) If $y = (\sin^{-1} x)^2$, prove that $(1 - x^2) y_2 - x y_1 = 2$

(*ii*) If $y = e^{m \sin^{-1} x}$, then $(1 + x^2) y_2 - x y_1 = m^2 y$

(*iii*) If $y = e^{\tan^{-1} x}$, then $(1 + x^2) y_2 + (2x - 1) y_1 = 0$

8. If $y = \log(1 + \cos x)$, prove that $y_1 y_2 + y_3 = 0$.

9. If $y = \dfrac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left(\sqrt{\dfrac{a - b}{a + b}} \tan\dfrac{x}{2}\right)$, show that $\dfrac{d^2y}{dx^2} = \dfrac{b \sin x}{(a + b\cos x)^2}$.

10. If $2y = x\left(1 + \dfrac{dy}{dx}\right)$, show that $\dfrac{d^2y}{dx^2}$ is constant.

11. If $x = \log t$, $t > 0$ and $y = \dfrac{1}{t}$ show that $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$.

12. Find $\dfrac{d^2y}{dx^2}$ when

(*i*) $x = a t^2$, $y = 2at$ (*ii*) $x = a\cos\theta$, $y = b\sin\theta$

(*iii*) $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$

(*iv*) $x = \cos\theta + \theta\sin\theta$, $y = \sin\theta - \theta\cos\theta$

(*v*) $x = a\left(\cos t + \log\tan\dfrac{t}{2}\right)$, $y = a\sin t$.

13. (*a*) If $x = 2\cos\theta - \cos 2\theta$, $y = 2\sin\theta - \sin 2\theta$, find the value of

$\dfrac{d^2y}{dx^2}$ at $\theta = \dfrac{\pi}{2}$.

(*b*) If $x = a\cos^3\theta$, $y = b\sin^3\theta$, find the value of $\dfrac{d^2y}{dx^2}$ at $\theta = \dfrac{\pi}{4}$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

14. (*i*) If $y^2 = ax^2 + b$, show that $\dfrac{d^2y}{dx^2} = \dfrac{ab}{y^3}$

(*ii*) If $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, prove that $\dfrac{d^2y}{dx^2} = -\dfrac{b^4}{a^2y^3}$

(*iii*) If $ax^2 + 2hxy + by^2 = 1$, prove that $\dfrac{d^2y}{dx^2} = \dfrac{h^2 - ab}{(hx + by)^3}$

(*iv*) If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, prove that

$$\frac{d^2y}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^3}$$

(*v*) If $x^3 + y^3 = 3axy$, prove that $\dfrac{d^2y}{dx^2} = -\dfrac{2a^3xy}{(y^2 - ax)^3}$.

15. Find a, b, c so that the function

$$f(x) = \begin{cases} x^3 + 2x + 3, & x \le 1 \\ ax^2 + bx + c, & x > 1 \end{cases}$$

is twice differentiable at $x = 1$.

16. For what choices of α, β and γ does the function

$$f(x) = \begin{cases} x^3, & x \le 1 \\ \alpha x^2 + \beta x + \gamma, & x > 1 \end{cases}$$

have a second order derivatives at $x = 1$?

17. Let $f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

Show that $\dfrac{d^2y}{dx^2}$ does not exist at $x = 0$.

18. If the function f is defined as

$$f(x) = \begin{cases} -x^2, & x \le 0 \\ x^2, & x > 0 \end{cases}$$

then the domain of f' is the set of all real numbers whereas the domain of f'' is the set of all real numbers except zero.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

ANSWERS

1. (*a*) (*i*) $20x^3 - 6x + \sqrt{2}$, $60x^2 - 6$, $120x$, 120, $f^{(n)}(x) = 0$ for $n \geq 5$

(*ii*) $4x^3 + 3x^2 + 6x + 7$, $12x^2 + 6x + 6$, $24x + 6$, 24, $f^{(n)}(x) = 0$ for $n \geq 5$

12. (*i*) $-\dfrac{1}{2at^3}$ (*ii*) $-\dfrac{b}{a^2}\operatorname{cosec}^3\theta$

(*iii*) $-\dfrac{1}{4a}\operatorname{cosec}^4\dfrac{\theta}{2}$ (*iv*) $\dfrac{\sec^3\theta}{\theta}$ (*v*) $\dfrac{1}{a}\dfrac{\sin t}{\cos^4 t}$

13. (*a*) $-\dfrac{3}{2}$ (*b*) $\dfrac{4\sqrt{2}\,b}{3a^2}$

15. $a = 3, b = -1, c = 4$ **16.** $\alpha = 3, \beta = -3, \gamma = 1$

Art-2. Derivatives of *n*th Order of Some Standard Functions

(*i*) If $y = (ax+b)^m$, then $y_n = m(m-1)(m-2)\ldots(m-n+1)(ax+b)^{m-n}a^n$.

Proof. Here $y = (ax+b)^m$

Differentiating both sides *w.r.t.* x, we get,

$y_1 = m(ax+b)^{m-1}.a = m(ax+b)^{m-1}.a^1$

$\therefore$ result is true for $n = 1$

Assume that the result is true for $n = k$, where k is positive integer.

$\therefore \quad y_k = m(m-1)(m-2)\ldots(m-k+1)(ax+b)^{m-k}.a^k$

Differentiating both sides w.r.t. x, we get,

$$y_{k+1} = m(m-1)(m-2)\ldots(m-k+1)(m-k)(ax+b)^{m-k-1}.a.a^k$$

or $\quad y_{k+1} = m(m-1)(m-2)\ldots(m-k+1)(m-k)(ax+b)^{m-(k+1)}.a^{k+1}$

$\therefore$ result is true for $n = k+1$.

$\therefore$ if the result is true for any positive integer k, then it is also true for the next higher integer $k+1$.

But the result is true for $n = 1$

$\therefore$ by method of induction, the result is true for all positive integers n.

Cor. 1. If m is a positive integer $> n$, then

$$y_n = \frac{m(m-1)(m-2)\ldots(m-n+1)\lfloor m-n}{\lfloor m-n}(ax+b)^{m-n}.a^n$$

or $$y_n = \frac{\lfloor m}{\lfloor m-n}(ax+b)^{m-n}.a^n$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

If $m = n$, then $y_n = n(n-1)(n-2)\ldots 2\cdot 1\,(ax+b)^0 \,.\, a^n$

or $y_n = \lfloor n \,.\, a^n$

$y_{n+1} = y_{n+2} = \ldots = 0$

$\therefore \quad y_n = 0 \;\; \forall \; n > m.$

(*ii*) If $y = \dfrac{1}{ax+b}$, then $y_n = \dfrac{(-1)^n \lfloor n \,.\, a^n}{(ax+b)^{n+1}}$, $x \neq -\dfrac{b}{a}$.

Proof. Here $y = \dfrac{1}{ax+b} = (ax+b)^{-1}$

$$\therefore \quad y_1 = (-1)(ax+b)^{-2} \,.\, a = \frac{(-1)^{-1} \,.\, \lfloor 1 \,.\, a^1}{(ax+b)^2}$$

$\therefore$ the result is true for $n = 1$

Assume that the result is true for $n = k$, where k is a positive integer.

$$\therefore \quad y_k = \frac{(-1)^k \,.\, \lfloor k \,.\, a^k}{(ax+b)^{k+1}} = (-1)^k \lfloor k \;\; a^k (ax+b)^{-k-1}$$

Differentiating again *w.r.t.* x, we get,

$$y_{k+1} = (-1)^k \lfloor k \;\; a^k (-k-1)(ax+b)^{-k-2} \,.\, a$$

$$= (-1)^{k+1} \lfloor k+1 \;\; a^{k+1} (ax+b)^{-(k+2)} = \frac{(-1)^{k+1} \lfloor k+1 \,.\, a^{k+1}}{(ax+b)^{k+2}}$$

$\therefore$ result is true for $n = k+1$.

$\therefore$ if the result is true for $n = k$, then it is also true for $n = k+1$.

But the result is true for $n = 1$.

$\therefore$ by method of induction, the result is true for all positive integers n.

(*iii*) If $y = \log(ax+b)$, then $y_n = \dfrac{(-1)^{n-1} \lfloor n-1 \,.\, a^n}{(ax+b)^n}$, $x > -\dfrac{b}{a}$.

Proof. Here $y = \log(ax+b)$

Differentiating both sides w.r.t. x,

$$y_1 = \frac{1}{ax+b} \,.\, a = \frac{(-1)^{1-1} \,.\, \lfloor 1-1 \,.\, a^1}{(ax+b)^1}$$

$\therefore$ the result is true for $n = 1$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Assume that the result is true for $n = k$, where k is a positive integer.

$$\therefore \quad y_k = \frac{(-1)^{k-1} . \lfloor k-1 . a^k}{(ax+b)^k} = (-1)^{k-1} \lfloor k-1 . a^k . (ax+b)^{-k}$$

Differentiating both sides w.r.t. x, we get,

$$y_{k+1} = (-1)^{k-1} \lfloor k-1 \; a^k (-k)(ax+b)^{-k-1} . a$$

$$= (-1)^k \lfloor k \; a^{k+1} (ax+b)^{-(k+1)} = \frac{(-1)^k \lfloor k . a^{k+1}}{(ax+b)^{k+1}}$$

$\therefore$ result is true for $n = k + 1$

$\therefore$ if the result is true for $n = k$, then it is also true for $n = k + 1$

But the result is true for $n = 1$.

$\therefore$ by the method of induction, the result is true for all positive integers n.

Note. Same result will hold even if $y = \log |ax + b|$ where $x > -\frac{b}{a}$.

(*iv*) If $y = a^{mx}$, $a > 0$, then $y_n = a^{mx} . (\log a)^n . m^n$.

Proof. Here $y = a^{mx}$

Differentiating both sides w.r.t. x,

$$y_1 = a^{mx} . \log a . m = a^{mx} . (\log a)^1 . m^1$$

$\therefore$ the result is true for $n = 1$.

Assume that the result is true for $n = k$, where k is a positive integer.

$$\therefore \quad y_k = a^{mx} . (\log a)^k . m^k$$

Differentiating both sides *w.r.t.* x,

$$y_{k+1} = [a^{mx} . (\log a) . (m)] . (\log a)^k . m^k = a^{mx} . (\log a)^{k+1} . m^{k+1}$$

$\therefore$ result is true for $n = k + 1$.

$\therefore$ if the result is true for $n = k$, then it is also true for $n = k + 1$.

But the result is true for $n = 1$.

$\therefore$ by the method of induction, the result is true for all positive integers n.

Cor. 1. Put $m = 1$

$$\therefore \quad y_n = a^x . (\log a)^n$$

$$\therefore \quad y = a^x \quad \Rightarrow \quad y_n = a^x . (\log a)^n$$

Cor. 2. Put $a = e$

$$\therefore \quad y_n = e^{mx} . (\log e)^n . m^n = e^{mx} . m^n$$

$$\therefore \quad y = e^{mx} \quad \Rightarrow \quad y_n = e^{mx} . m^n$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Cor. 3. Put $a = e,\ m = 1$

$$\therefore \quad y_n = e^x . (\log e)^n . (1)^n = e^x$$

$$\therefore \quad y = e^x \quad \Rightarrow \quad y_n = e^x.$$

(*v*) If $y = \sin(ax + b)$, then $y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right) \ \forall\ x \in \mathbf{R}$.

Proof. Here $y = \sin(ax + b)$

Differentiating both sides w.r.t. x,

$$y_1 = \cos(ax + b) . a = a^1 \sin\left[ax + b + 1.\frac{\pi}{2}\right]$$

$\therefore$ the result is true for $n = 1$.

Assume that the result is true for $n = k$, where k is a positive integer.

$$\therefore \quad y_k = a^k \sin\left[ax + b + k\frac{\pi}{2}\right]$$

Differentiating again w.r.t. x,

$$y_{k+1} = a^k \cos\left[ax + b + k\frac{\pi}{2}\right] . a = a^{k+1} \sin\left[\left(ax + b + k\frac{\pi}{2}\right) + \frac{\pi}{2}\right]$$

$$= a^{k+1} \sin\left[ax + b + (k+1)\frac{\pi}{2}\right]$$

$\therefore$ result is true for $n = k + 1$.

$\therefore$ if the result is true for $n = k$, then it is also true for $n = k + 1$.

But the result is true for $n = 1$.

$\therefore$ by the method of induction, the result is true for all positive integers n.

(*vi*) If $y = \cos(ax + b)$, then $y_n = a^n \cos\left(ax + b + n\frac{\pi}{2}\right) \ \forall\ x \in \mathbf{R}$.

Proof. Here $y = \cos(ax + b)$

Differentiating both sides *w.r.t.* x,

$$y_1 = -\sin(ax + b) . a = a^1 . \cos\left[ax + b + 1.\frac{\pi}{2}\right]$$

$\therefore$ the result is true for $n = 1$.

Assume that the result is true for $n = k$, where k is a positive integer.

$$\therefore \quad y_k = a^k \cos\left[ax + b + k\frac{\pi}{2}\right]$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Differentiating again *w.r.t.* x, we get,

$$y_{k+1} = a^k \left[-\sin\left(ax + b + k\frac{\pi}{2}\right).a \right] = a^{k+1} \cos\left[\left(ax + b + k\frac{\pi}{2}\right) + \frac{\pi}{2}\right]$$

$$= a^{k+1} \cos\left[ax + b + (k+1)\frac{\pi}{2}\right]$$

$\therefore$ the result is true for $n = k + 1$.

$\therefore$ if the result is true for $n = k$, then it is also true for $n = k + 1$.

But the result is true for $n = 1$

$\therefore$ by the method of induction, the result is true for all positive integers n.

(*vii*) If $y = e^{ax} \sin(bx + c)$, then $y_n = (a^2 + b^2)^{\frac{n}{2}} e^{ax} \sin\left(bx + c + n\tan^{-1}\frac{b}{a}\right)$

Proof. Here $y = e^{ax} \sin(bx + c)$

Differentiating both sides *w.r.t.* x,

$$y_1 = e^{ax} . \frac{d}{dx}[\sin(bx + c)] + \sin(bx + c) . \frac{d}{dx}(e^{ax})$$

$$= e^{ax} . \cos(bx + c) . b + \sin(bx + c) . e^{ax} . a$$

$\therefore$ $y_1 = e^{ax}[a \sin(bx + c) + b \cos(bx + c)]$ ...(1)

Put $a = r\cos\alpha$ ...(2)

$b = r\sin\alpha$ where $r > 0$. ...(3)

Squaring and adding (2) and (3), we get,

$$a^2 + b^2 = r^2 \Rightarrow r = \sqrt{a^2 + b^2}$$

Dividing (3) by (2), $\tan\alpha = \frac{b}{a} \Rightarrow \alpha = \tan^{-1}\frac{b}{a}$

$\therefore$ from (1), $y_1 = e^{ax}[r\cos\alpha \sin(bx + c) + r\sin\alpha \cos(bx + c)]$

$$= e^{ax} . r[\sin(bx + c)\cos\alpha + \cos(bx + c) . \sin\alpha]$$

$$= r e^{ax} \sin(bx + c + \alpha)$$

$$\therefore \quad y_1 = (a^2 + b^2)^{\frac{1}{2}} . e^{ax} \sin\left(bx + c + 1.\tan^{-1}\frac{b}{a}\right)$$

$\therefore$ the result is true for $n = 1$.

Assume that the result is true for $n = k$, where k is a positive integer.

$$\therefore \quad y_k = (a^2 + b^2)^{\frac{k}{2}} . e^{ax} \sin\left(bx + c + k\tan^{-1}\frac{b}{a}\right)$$

or $y_k = r^k e^{ax} \sin(bx + c + k\alpha)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Differentiating again *w.r.t.* x, we get,

$$y_{k+1} = r^k . [e^{ax} . \cos(bx + c + k\alpha) . b + \sin(bx + c + k\alpha) . a e^{ax}]$$

$$= r^k . e^{ax} [a \sin(bx + c + k\alpha) + b \cos(bx + c + k\alpha)]$$

$$= r^k . e^{ax} [r \cos\alpha \sin(bx + c + k\alpha) + r \sin\alpha \cos(bx + c + k\alpha)]$$

$$= r^{k+1} . e^{ax} \sin[(bx + c + k\alpha) + \alpha] = r^{k+1} . e^{ax} \sin[bx + c + (k+1)\alpha]$$

$$\therefore \quad y_{k+1} = (a^2 + b^2)^{\frac{k+1}{2}} . e^{ax} \sin\left(bx + c + (k+1)\tan^{-1}\frac{b}{a}\right)$$

$\therefore$ the result is true for $n = k + 1$

$\therefore$ if the result is true for $n = k$, then it is also true for $n = k + 1$.

But the result is true for $n = 1$.

$\therefore$ by the method of induction, the result is true for all positive integers n.

(*viii*) If $y = e^{ax} \cos(bx + c)$, then

$$y_n = (a^2 + b^2)^{\frac{n}{2}} . e^{ax} \cos\left(bx + c + n\tan^{-1}\frac{b}{a}\right).$$

Proof. Here $y = e^{ax} \cos(bx + c)$

Differentiating *w.r.t.* x,

$$y_1 = e^{ax} \frac{d}{dx} [\cos(bx + c)] + \cos(bx + c) . \frac{d}{dx} (e^{ax})$$

$$= e^{ax} . [-\sin(bx + c) . b] + \cos(bx + c) . (e^{ax} . a)$$

$$\therefore \quad y_1 = e^{ax} [a \cos(bx + c) - b \sin(bx + c)] \quad \ldots(1)$$

Put $a = r \cos\alpha$ ...(2)

$b = r \sin\alpha$ where $r > 0$. ...(3)

Squaring and adding (2) and (3), we get,

$$a^2 + b^2 = r^2 \Rightarrow r = \sqrt{a^2 + b^2}$$

Dividing (3) by (2), $\tan\alpha = \dfrac{b}{a} \Rightarrow \alpha = \tan^{-1}\dfrac{b}{a}$

$\therefore$ from (1), $y_1 = e^{ax} [r \cos\alpha \cos(bx + c) - r \sin\alpha \sin(bx + c)]$

$$= r . e^{ax} [\cos(bx + c) \cos\alpha - \sin(bx + c) . \sin\alpha]$$

$$= r e^{ax} \cos(bx + c + \alpha)$$

$$\therefore \quad y_1 = (a^2 + b^2)^{\frac{1}{2}} . e^{ax} \cos\left(bx + c + 1 . \tan^{-1}\frac{b}{a}\right)$$

$\therefore$ the result is true for $n = 1$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Assume that the result is true for $n = k$, where k is a positive integer.

$$\therefore \quad y_k = (a^2 + b^2)^{\frac{k}{2}} . e^{ax} \cos\left(bx + c + k \tan^{-1}\frac{b}{a}\right)$$

or $\quad y_k = r^k e^{ax} \cos(bx + c + k\alpha)$

Differentiating again *w.r.t.* x, we get,

$$y_{k+1} = r^k . [e^{ax} . \{-\sin(bx + c + k\alpha) . b\} + \cos(bx + c + k\alpha) . a e^{ax}]$$

$$= r^k e^{ax} [a \cos(bx + c + k\alpha) - b \sin(bx + c + k\alpha)]$$

$$= r^k e^{ax} [r \cos\alpha \cos(bx + c + k\alpha) - r \sin\alpha \sin(bx + c + k\alpha)]$$

$$= r^{k+1} e^{ax} [\cos(bx + c + k\alpha) . \cos\alpha - \sin(bx + c + k\alpha) . \sin\alpha]$$

$$= r^{k+1} . e^{ax} \cos[(bx + c + k\alpha) + \alpha]$$

$$\therefore \quad y_{k+1} = r^{k+1} . e^{ax} \cos[bx + c + (k+1)\alpha]$$

or $\quad y_{k+1} = (a^2 + b^2)^{\frac{k+1}{2}} . e^{ax} \cos\left(bx + c + (k+1)\tan^{-1}\frac{b}{a}\right)$

$\therefore$ the result is true for $n = k + 1$

$\therefore$ if the result is true for $n = k$, then it is also true for $n = k + 1$.

But the result is true for $n = 1$.

$\therefore$ by the method of induction, the result is true for all positive integers n.

ILLUSTRATIVE EXAMPLES

Example 1. Prove that $\dfrac{d^n}{dx^n}\left(\dfrac{\log x}{x}\right) = \dfrac{(-1)^n \lfloor n}{x^{n+1}}\left[\log x - 1 - \dfrac{1}{2} - \dfrac{1}{3} \ldots - \dfrac{1}{n}\right]$.

Sol. Let $y = \dfrac{\log x}{x}$

Differentiating *w.r.t.* x, we get,

$$y_1 = \frac{x . \frac{1}{x} - \log x . 1}{(x)^2} = \frac{1 - \log x}{x^2} = \frac{(-1)^1 \lfloor 1}{x^2} [\log x - 1]$$

$\therefore$ the result is true for $n = 1$

Assume that the result is true for $n = k$.

$$\therefore \quad y_k = \frac{(-1)^k \lfloor k}{x^{k+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{k}\right]$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Differentiating w.r.t. x, we get,

$$y_{k+1} = (-1)^k \lfloor k \frac{d}{dx}\left[\frac{1}{x^{k+1}}.\left\{\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{k}\right\}\right]$$

$$= (-1)^k \lfloor k \left[\frac{1}{x^{k+1}}\frac{d}{dx}\left\{\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{k}\right\}\right.$$

$$\left. + \left\{\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{k}\right\}\frac{d}{dx}(x^{-k-1})\right]$$

$$= (-1)^k \lfloor k \left[\frac{1}{x^{k+1}}.\frac{1}{x} + \left\{\log x - 1 - \frac{1}{2} - \frac{1}{k}\right\}(-k-1)x^{-k-2}\right]$$

$$= (-1)^k \lfloor k \left[\frac{1}{x^{k+2}} - \frac{k+1}{x^{k+2}} + \left\{\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{k}\right\}\right]$$

$$= \frac{(-1)^{k+1}(k+1)\lfloor k}{x^{k+2}}\left[-\frac{1}{k+1} + \left\{\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{k}\right\}\right]$$

$$= \frac{(-1)^{k+1}\lfloor k+1}{x^{k+2}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{k} - \frac{1}{k+1}\right]$$

$\therefore$ the result is true for $n = k + 1$

$\therefore$ if the result is true for $n = k$, then it is also true for $n = k + 1$

But the result is true for $n = 1$.

$\therefore$ by the method of induction, the result is true for all positive integers n.

$$\therefore \quad y_n = \frac{(-1)^n \lfloor n}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{n}\right]$$

$$\text{or} \quad \frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{(-1)^n \lfloor n}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} \ldots - \frac{1}{n}\right].$$

Example 2. Find the nth derivative of $\sqrt{ax+b}$.

Sol. Let $y = \sqrt{ax+b} = (ax+b)^{\frac{1}{2}}$

$$\therefore \quad y_1 = \frac{1}{2}(ax+b)^{\frac{1}{2}-1}.a = \frac{1}{2}(ax+b)^{-\frac{1}{2}}\, a$$

$$\therefore \quad y_2 = \frac{1}{2}\left(-\frac{1}{2}\right)(ax+b)^{-\frac{3}{2}}\, a^2$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\therefore \quad y_3 = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right).(ax+b)^{-\frac{5}{2}}\, a^3$$

$$\therefore \quad y_3 = \frac{(-1)^2\, 1.3}{2^3}\,(ax+b)^{\frac{1}{2}-3}\, a^3$$

.....

$$\therefore \quad y_n = \frac{(-1)^{n-1}\, 1.3.5\ldots(2n-1)\,(ax+b)^{\frac{1}{2}-n}\, a^n}{2^n}$$

$$\therefore \quad y_n = \frac{(-1)^{n-1}\, 1.3.5\ldots(2n-1)}{2^n\,(ax+b)^{\frac{2n-1}{2}}}.a^n \quad \text{where } x \neq -\frac{b}{a}$$

Example 3. If $y = a^{2x} + \dfrac{x}{2x-1}$, find y_n.

Sol. Here $y = a^{2x} + \dfrac{x}{2x-1}$

$$\therefore \quad y = a^{2x} + \frac{1}{2} + \frac{1}{2(2x-1)}$$

$$\therefore \quad y_n = a^{2x}(2\log a)^n + 0 + \frac{1}{2}\,\frac{(-1)^n \lfloor n\; 2^n}{(2x-1)^{n+1}}$$

$$\begin{array}{r|l} & \frac{1}{2} \\ \hline 2x-1 & x \\ & x \quad -\frac{1}{2} \\ & - \quad + \\ \hline & \frac{1}{2} \end{array}$$

or $\quad y_n = a^{2x}(2\log a)^n + \dfrac{(-1)^n \lfloor n\; 2^{n-1}}{(2x-1)^{n+1}}$.

Example 4. Find the nth derivative of $\dfrac{x^2}{(x+2)(2x+3)}$.

Sol. Let $y = \dfrac{x^2}{(x+2)(2x+3)} = \dfrac{1}{2} + \dfrac{(-2)^2}{(x+2)(-4+3)} + \dfrac{\left(-\frac{3}{2}\right)^2}{\left(-\frac{3}{2}+2\right)(2x+3)}$

$$= \frac{1}{2} - \frac{4}{x+2} + \frac{9}{2(2x+3)} = \frac{1}{2}\left[1 - \frac{8}{x+2} + \frac{9}{2x+3}\right]$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\therefore \quad y_n = \frac{1}{2}\left[0 - 8\frac{(-1)^n \lfloor n}{(x+2)^{n+1}} + 9.\frac{(-1)^n.\lfloor n.2^n}{(2x+3)^{n+1}}\right]$$

$$= \frac{(-1)^n \lfloor n}{2}\left[\frac{9\times 2^n}{(2x+3)^{n+1}} - \frac{8}{(x+2)^{n+1}}\right].$$

Example 5. Find y_n, if $y = \dfrac{2x+1}{(x-2)(x-1)^3}$.

Sol.
$$y = \frac{2x+1}{(x-2)(x-1)^3} \qquad \ldots(1)$$

Put $\dfrac{2x+1}{(x-2)(x-1)^3} \equiv \dfrac{A}{x-2} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2} + \dfrac{D}{(x-1)^3}$

Multiplying both sides by $(x-2)(x-1)^3$, we get

$$2x+1 \equiv A(x-1)^3 + B(x-2)(x-1)^2 + C(x-2)(x-1) + D(x-2) \quad \ldots(1)$$

Putting $x-2=0$ *i.e.* $x=2$ in (1), we get

$$5 = A \Rightarrow A = 5$$

Putting $x-1=0$ *i.e.* $x=1$ in (1), we get

$$3 = -D \Rightarrow D = -3$$

(1) can be writing as

$$2x+1 \equiv A(x^3-3x^2+3x-1) + B(x^3-4x^2+5x-2) + C(x^2-3x+2) + D(x-2) \qquad \ldots(2)$$

Equating coefficients in (2) of

x^3) $\quad A + B = 0 \Rightarrow 5 + B = 0 \Rightarrow B = -5$

x^2) $\quad -3A - 4B + C = 0 \Rightarrow -15 + 20 + C = 0 \Rightarrow C = -5$

$$\therefore \quad \frac{2x+1}{(x-2)(x-1)^2} \equiv \frac{5}{x-2} - \frac{5}{x-1} - \frac{5}{(x-1)^2} - \frac{3}{(x-1)^3}$$

$$\therefore \quad y = \frac{5}{x-2} - \frac{5}{x-1} - \frac{5}{(x-1)^2} - \frac{3}{(x-1)^3}$$

$$\therefore \quad y_n = 5\left[\frac{(-1)^n \lfloor n}{(x-2)^{n+1}}\right] - 5\left[\frac{(-1)^n \lfloor n}{(x-1)^{n+1}}\right] - 5\left[\frac{(-1)^n \lfloor n+1}{(x-1)^{n+2}}\right] - 3\left[\frac{(-1)^n \lfloor n+2}{(x-1)^{n+3}}\right].$$

$$\therefore \quad y_n = (-1)^n \lfloor n \left[\frac{5}{(x-2)^{n+1}} - \frac{5}{(x-1)^{n+1}} - \frac{5(n+1)}{(x-1)^{n+2}} - \frac{3(n+2)(n+1)}{(x-1)^{n+3}}\right]$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 6. If $y = \sin^2 x \cos^3 x$, find y_n.

Sol. Here $y = \sin^2 x \cos^3 x$

$$\therefore \quad y = \frac{1}{4}(2\sin x \cos x)^2 \cos x = \frac{1}{4}\sin^2 2x \,.\, \cos x$$

$$= \frac{1}{4}.\frac{1-\cos 4x}{2}\,.\cos x = \frac{1}{8}[\cos x - \cos 4x \cos x]$$

$$= \frac{1}{8}\left[\cos x - \frac{\cos 5x + \cos 3x}{2}\right] = \frac{1}{8}\cos x - \frac{1}{16}\cos 3x - \frac{1}{16}\cos 5x$$

Differentiating n times, we get,

$$y_n = \frac{1}{8}\cos\left(x + \frac{n\pi}{2}\right) - \frac{1}{16}.3^n \cos\left(3x + \frac{n\pi}{2}\right) - \frac{1}{16}.5^n \cos\left(5x + \frac{n\pi}{2}\right).$$

Example 7. Find the nth derivative of $e^x \cos^3 x$.

Sol. We know $\cos^3 x = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x$

$$\therefore \quad y = e^x \cos^3 x = e^x\left[\frac{1}{4}\cos 3x + \frac{3}{4}\cos x\right]$$

$$\therefore \quad y = \frac{1}{4}e^x \cos 3x + \frac{3}{4}e^x \cos x$$

$$\therefore \quad y_n = \frac{1}{4}(1^2 + 3^2)^{\frac{n}{2}} e^x \cos\left(3x + n\tan^{-1}\frac{3}{1}\right) + \frac{3}{4}[1^2 + 1^2]^{\frac{n}{2}} e^x \cos\left(x + n\tan^{-1}\frac{1}{1}\right)$$

$$= \frac{1}{4}(10)^{\frac{n}{2}} e^x \cos(3x + n\tan^{-1} 3) + \frac{3}{4}(2)^{\frac{n}{2}} e^x \cos\left(x + \frac{n\pi}{4}\right).$$

Example 8. Find the nth derivative of $y = e^{3x}\sin^2 2x$.

Sol.
$$y = e^{3x}\sin^2 2x = e^{3x}\left(\frac{1-\cos 4x}{2}\right) = \frac{1}{2}e^{3x} - \frac{1}{2}e^{3x}\cos 4x$$

$$\therefore \quad y_n = \frac{1}{2}e^{3x}.3^n - \frac{1}{2}\left[(9+16)^{\frac{n}{2}} e^{3x}\cos\left(4x + n\tan^{-1}\frac{4}{3}\right)\right]$$

$$\therefore \quad y_n = \frac{1}{2}e^{3x}\left[3^n - 5^n\cos\left(4x + n\tan^{-1}\frac{4}{3}\right)\right]$$

Example 9. Prove that $y_n(0) = 0$ if n is an even + ve integer and $y_n(0) = -\lfloor n$ if n if an odd + ve integer > 1where $y = \dfrac{x^3}{x^2 - 1}$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Sol. Here $y = \dfrac{x^3}{x^2-1} = x + \dfrac{x}{x^2-1} = x + \dfrac{x}{(x-1)(x+1)}$

$$= x + \frac{1}{(x-1)(1+1)} + \frac{-1}{(-1-1)(x+1)} = x + \frac{1}{2}\left(\frac{1}{x-1} + \frac{1}{x+1}\right)$$

$$\therefore \quad y_n = 0 + \frac{1}{2}\left[\frac{(-1)^n \lfloor n}{(x-1)^{n+1}} + \frac{(-1)^n \lfloor n}{(x+1)^{n+1}}\right]$$

$$\therefore \quad y_n(0) = \frac{1}{2}(-1)^n \lfloor n \left[\frac{1}{(-1)^{n+1}} + \frac{1}{(1)^{n+1}}\right]$$

Two cases arise :

Case (*i*) When n is a + ve even integer

$$y_n(0) = \frac{1}{2}[1] \lfloor n \left[\frac{1}{-1} + \frac{1}{1}\right] = \frac{1}{2} \times 1 \times \lfloor n \times 0 = 0$$

$$\therefore \quad y_n(0) = 0$$

Case (*ii*) When n is an odd + ve integer.

$$y_n(0) = \frac{1}{2}(-1) \lfloor n \left[\frac{1}{1} + \frac{1}{1}\right] = \frac{1}{2}(-1) \lfloor n \ (2) = -\lfloor n$$

Hence the result.

Example 10. Find y_n if $y = \dfrac{1}{x^2 + a^2}$.

Sol. Let $y = \dfrac{1}{x^2 + a^2}$

Put $x = a \cot\theta$ so that $\dfrac{dx}{d\theta} = -a \operatorname{cosec}^2\theta = -\dfrac{a}{\sin^2 6}$

$$\therefore \quad y = \frac{1}{a^2 \cot^2 6 + a^2} = \frac{1}{a^2 \operatorname{cosec}^2 6} = \frac{1}{a^2} \cdot \sin^2\theta$$

Differentiating w.r.t. x, we get,

$$\frac{dy}{dx} = \frac{1}{a^2} 2\sin\theta\cos\theta \frac{d\theta}{dx} = \frac{1}{a^2} 2\sin\theta\cos\theta \frac{\sin^2\theta}{-a}$$

$$\frac{dy}{dx} = -\frac{1}{a^3} \sin 2\theta \sin^2\theta$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Differentiating w.r.t. x,

$$\frac{d^2y}{dx^2} = -\frac{1}{a^3}\,[2\cos 2\theta \sin^2\theta + \sin 2\theta \,.\, 2\sin\theta\cos\theta\,]\,\frac{d\theta}{dx}$$

$$= -\frac{1}{a^3}\,[2\sin\theta\,]\,[\cos 2\theta\sin\theta + \sin 2\theta\cos\theta\,]\,\frac{\sin^2\theta}{-a}$$

$$= (-1)^2\,\frac{1}{a^4}\,2\sin\theta\sin(2\theta+\theta)\sin^2\theta$$

$$= (-1)^2\,\frac{\lfloor 2}{a^4}\,\sin^3\theta\sin 3\theta$$

Similarly $\dfrac{d^3y}{dx^3} = (-1)^3\,\dfrac{\lfloor 3}{a^5}\,\sin^4\theta\sin 4\theta$ and so on

$\therefore$ in general $\dfrac{d^ny}{dx^n} = (-1)^n\,\dfrac{1}{a^{n+2}}\lfloor n\,.\,\sin^{n+1}\theta\sin(n+1)\theta$

where $\theta = \cot^{-1}\dfrac{x}{a}$ or $\theta = \tan^{-1}\dfrac{a}{x}$.

Example 11. Find the nth derivative of $\dfrac{x}{x^2+a^2}$.

Sol. $$y = \frac{x}{x^2+a^2} = \frac{x}{x^2 - i^2a^2}$$

$$\therefore \quad y = \frac{x}{(x-ai)(x+ai)} = \frac{1}{2}\left[\frac{1}{(x-ai)} + \frac{1}{(x+ai)}\right]$$

$$\therefore \quad \frac{d^ny}{dx^n} = \frac{(-1)^n\lfloor n}{2}\left[\frac{1}{(x-ai)^{n+1}} + \frac{1}{(x+ai)^{n+1}}\right] \qquad \ldots(1)$$

Now $\dfrac{1}{(x-ai)^{n+1}}$

[Put $x = r\cos\theta$, $a = r\sin\theta$ Squaring and adding, $r = \sqrt{x^2+a^2}$

Dividing $\tan\theta = \dfrac{a}{x}$]

$$= \frac{1}{(r\cos\theta - ir\sin\theta)^{n+1}} = \frac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}}$$

$$= \frac{1}{r^{n+1}}\,(\cos\theta + i\sin\theta)^{n+1}$$

$$= \frac{1}{r^{n+1}}\,[\cos(n+1)\theta + i\sin(n+1)\theta]$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

and $$\frac{1}{(x+ai)^{n+1}} = \frac{1}{r^{n+1}}\,[\cos\theta + i\sin\theta]^{-(n+1)}$$

$$= \frac{1}{r^{n+1}}\,[\cos(n+1)\theta - i\sin(n+1)\theta]$$

$$\therefore \quad \frac{1}{(x-ai)^{n+1}} + \frac{1}{(x+ai)^{n+1}} = \frac{1}{r^{n+1}}\,2\cos(n+1)\theta$$

$\therefore$ from (1), we get,

$$\frac{d^n}{dx^n}\left(\frac{x}{x^2+a^2}\right) = \frac{1}{2}\frac{(-1)^n \lfloor n}{r^{n+1}}\,2\cos(n+1)\theta$$

$$= \frac{(-1)^n \lfloor n}{(x^2+a^2)^{\frac{n+1}{2}}}\cos\left[(n+1)\tan^{-1}\left(\frac{a}{x}\right)\right].$$

Example 12. If $y = x\log\left(\frac{x-1}{x+1}\right)$, prove that

$$\frac{d^n y}{dx^y} = (-1)^n \lfloor n-2 \left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right].$$

Sol. $y = x\log\left(\frac{x-1}{x+1}\right) = x\{\log(x-1) - \log(x+1)\}$

$$\therefore \quad y_1 = x\left\{\frac{1}{x-1} - \frac{1}{x+1}\right\} + \{\log(x-1) - \log(x+1)\}.\,1$$

$$= \frac{x}{x-1} - \frac{x}{x+1} + \log(x-1) - \log(x+1)$$

$$= \left(1 + \frac{1}{x-1}\right) - \left(1 - \frac{1}{x+1}\right) + \log(x-1) - \log(x+1)$$

$$y_1 = \frac{1}{x-1} + \frac{1}{x+1} + \log(x-1) - \log(x+1)$$

Differentiating $(n-1)$ times *w.r.t.* x, we get,

$$y_n = \frac{(-1)^{n-1}\lfloor n-1}{(x-1)^n} + \frac{(-1)^{n-1}\lfloor n-1}{(x+1)^n} + \frac{(-1)^{n-2}\lfloor n-2}{(x-1)^{n-1}} - \frac{(-1)^{n-2}\lfloor n-2}{(x+1)^{n-1}}$$

$$= \left[\frac{(-1)^{n-1}\lfloor n-1}{(x-1)^n} + \frac{(-1)^{n-2}\lfloor n-2}{(x-1)^{n-1}}\right] + \left[\frac{(-1)^{n-1}\lfloor n-1}{(x+1)^n} - \frac{(-1)^{n-2}\lfloor n-2}{(x+1)^{n-1}}\right]$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$= (-1)^n \underline{|n-2} \left[\frac{-(n-1)+(x-1)}{(x-1)^{n-1}}\right] + (-1)^n \underline{|n-2} \left[\frac{-(n-1)-(x+1)}{(x+1)^{n+1}}\right]$$

$$= (-1)^n \underline{|n-2} \left[\frac{x-n}{(x-1)^{n-1}}\right] + (-1)^n \underline{|n-2} \left[-\frac{x+n}{(x+1)^{n+1}}\right]$$

$$= (-1)^n \underline{|n-2} \left[\frac{x-n}{(x-1)^{n-1}} - \frac{x+n}{(x+1)^{n+1}}\right].$$

EXERCISE 3 (*b*)

1. Find the *n*th derivative of

(*i*) $\sqrt{1-2x}$, $x < \frac{1}{2}$ (*ii*) $\sqrt{2-3x}$, $x < \frac{3}{2}$ (*iii*) $\frac{1}{1-2x}$, $x \neq \frac{1}{2}$

(*iv*) $\frac{3x+7}{x+2}$, $x \neq -2$ (*v*) $\frac{3}{(5+2x)^3}$, $x \neq -\frac{5}{2}$ (*vi*) $2^x . e^x$.

2. Find the *n*th derivative of

(*i*) $\log(x^2 - a^2)$ (*ii*) $\log(ax + x^2)$ (*iii*) $\log\left(\sqrt{\frac{1-2x}{5x+3}}\right)$

(*iv*) $\log\left(\sqrt{\frac{x-2}{2x+1}}\right)$ (*v*) $x \log\left(\frac{x-a}{x+a}\right)$, $x > a > 0$

3. Find the *n*th derivative of

(*i*) $\frac{1}{(x+2)(x+3)}$ (*ii*) $\frac{x}{1+3x+2x^2}$

(*iii*) $\frac{x+1}{x^2-4}$ (*iv*) $\frac{2x+1}{(x+2)(x-1)^2}$.

4. Find the *n*th derivative of

(*i*) $\sin^2 x$ (*ii*) $\sin^3 x$ (*iii*) $\cos^3 x$

(*iv*) $\sin^4 x$ (*v*) $\cos^4 x$ (*vi*) $\sin x \sin 2x$

(*vii*) $\sin 2x \sin 5x$ (*viii*) $\cos x \cos 2x \cos 3x$ (*ix*) $\sin^3 x \cos^2 x$

5. Find the *n*th derivative of

(*i*) $e^x \sin \sqrt{3}x$ (*ii*) $e^x \cos x \cos 2x$

(*iii*) $e^{ax} \cos^2 bx$ (*iv*) $e^{3x} \sin^2 x \cos^3 x$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

6. Show that the nth differential coefficient of

$$\frac{1}{1+x+x^2+x^3} \text{ is } \frac{1}{2}(-1)^n \lfloor n \sin^{n+1}\theta \left[\sin(n+1)\theta - \cos(n+1)\theta + (\sin\theta+\cos\theta)^{-n-1}\right]$$

where $\theta = \cot^{-1}(x)$.

7. Show that the nth derivative of $y = \tan^{-1} x$ is

$$(-1)^{n-1} \lfloor n-1 \sin n\left(\frac{\pi}{2}-y\right) \sin^n\left(\frac{\pi}{2}-y\right).$$

Also prove that this nth derivative can also be written as

$$\lfloor n-1 \cos\left[ny + (n-1)\frac{\pi}{2}\right] \cos^n y.$$

8. If $\mu = \sin nx + \cos nx$, prove that $\mu_r = n^r \left[1+(-1)^r \sin 2nx\right]^{\frac{1}{2}}$ where μ_r denotes the rth differential coefficient of μ w.r.t. x.

ANSWERS

1. (*i*) $y_n = (-1)^{2n-1} \dfrac{1.3.5.....(2n-3)}{(1-2x)^{\frac{2n-1}{2}}}, n \neq 1$; $y_1 = -\dfrac{1}{(1-2x)^{\frac{1}{2}}}$.

(*ii*) $y_n = -\dfrac{1.3.5...(2n-3)}{(2-3x)^{\frac{2n-1}{2}}}\left(\dfrac{3}{2}\right)^n, n \neq 1$; $y_1 = \dfrac{3}{2(2-3x)^{\frac{1}{2}}}$

(*iii*) $\dfrac{2^n \lfloor n}{(1-2x)^{n+1}}$ (*iv*) $\dfrac{(-1)^n \lfloor n}{(x+2)^{n+1}}$

(*v*) $(-1)^n \dfrac{3 \lfloor n+2 \, .2^{n-1}}{(5+2x)^{n+3}}$ (*vi*) $2^x e^x (1+\log 2)^n$

2. (*i*) $(-1)^{n-1} \lfloor n-1 \left[\dfrac{1}{(x+a)^n} + \dfrac{1}{(x-a)^n}\right]$

(*ii*) $(-1)^{n-1} \lfloor n-1 \left[\dfrac{1}{x^n} + \dfrac{1}{(x+a)^n}\right]$

(*iii*) $\dfrac{(-1)^{n-1} \lfloor n-1}{2}\left[\dfrac{(-1)^n .2^n}{(1-2x)^n} - \dfrac{5^n}{(5x+3)^n}\right]$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

(iv) $a\,(-1)^{n-1}\,\underline{|\,n-1}\left[\dfrac{1}{(x-a)^n}+\dfrac{1}{(x+a)^n}\right]$

$$+\,(-1)^{n-2}\,\underline{|\,n-2}\left[\frac{1}{(x-a)^{n-1}}-\frac{1}{(x+a)^{n-1}}\right]$$

3. (i) $(-1)^n\,\underline{|\,n}\left[\dfrac{1}{(x+2)^{n+1}}-\dfrac{1}{(x+3)^n}\right]$

(ii) $(-1)^n\,\underline{|\,n}\left[\dfrac{1}{(x+1)^{n+1}}-\dfrac{2^n}{(2x+1)^{n+1}}\right]$

(iii) $\dfrac{(-1)^n\,\underline{|\,n}}{4}\left[\dfrac{3}{(x-2)^{n+1}}+\dfrac{1}{(x+2)^{n+1}}\right]$

(iv) $-\dfrac{(-1)^n\,\underline{|\,n}}{3(x+2)^{n+1}}+\dfrac{(-1)^n\,\underline{|\,n+1}}{(x-1)^{n+2}}+\dfrac{(-1)^n\,\underline{|\,n}}{3(x-1)^{n+1}}$

4. (i) $-2^{n-1}\cos\left(2x+\dfrac{n\pi}{2}\right)$ (ii) $\dfrac{3}{4}\sin\left(x+\dfrac{n\pi}{2}\right)-\dfrac{1}{4}\,.\,3^n.\sin\left(3x+\dfrac{n\pi}{2}\right)$

(iii) $\dfrac{1}{4}\left[3^n\sin\left(3x+\dfrac{n\pi}{2}\right)+3\cos\left(x+\dfrac{n\pi}{2}\right)\right]$

(iv) $-\,2^{n-1}\cos\left(2x+\dfrac{n\pi}{2}\right)+2^{2n-3}\cos\left(4x+\dfrac{n\pi}{2}\right)$

(v) $2^{n-1}\cos\left(2x+\dfrac{n\pi}{2}\right)+2^{2n-3}\cos\left(4x+\dfrac{n\pi}{2}\right)$

(vi) $\dfrac{1}{2}\left[\cos\left(x+\dfrac{n\pi}{2}\right)-3^n\cos\left(3x+\dfrac{n\pi}{2}\right)\right]$

(vii) $\dfrac{1}{2}\left[3^n\cos\left(3x+\dfrac{n\pi}{2}\right)-7^n\cos\left(7x+\dfrac{n\pi}{2}\right)\right]$

(viii) $\dfrac{1}{4}\left[2^n\cos\left(2x+\dfrac{n\pi}{2}\right)+4^n\cos\left(4x+\dfrac{n\pi}{2}\right)+6^n\cos\left(6x+\dfrac{n\pi}{2}\right)\right]$

(ix) $\dfrac{1}{16}\left[2\sin\left(x+\dfrac{n\pi}{2}\right)-5^n\sin\left(5x+\dfrac{n\pi}{2}\right)+3^n\sin\left(3x+\dfrac{n\pi}{2}\right)\right]$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

5. (*i*) $2^n e^x \sin\left(\sqrt{3}\,x + \frac{n\pi}{2}\right)$

(*ii*) $\frac{1}{2} e^x \left[2^{\frac{n}{2}} \cos\left(x + \frac{n\pi}{4}\right) + 10^{\frac{n}{2}} \cos\left(3x + n\tan^{-1} 3\right)\right]$

(*iii*) $\frac{1}{2} e^{ax} \left[a^n + (a^2 + b^2)^{\frac{n}{2}} \cos\left(2bx + n\tan^{-1}\frac{2b}{a}\right)\right]$

(*iv*) $\frac{1}{16} e^{3x} \left[2.(10)^{\frac{n}{2}} \cos\left(x + n\tan^{-1}\frac{1}{3}\right)\right.$

$$\left. - (18)^{\frac{n}{2}} \cos\left(3x + \frac{n\pi}{4}\right) - (34)^{\frac{n}{2}} \cos\left(5x + n\tan^{-1}\frac{5}{3}\right)\right]$$

Art-3. Leibnitz's Theorem

This theorem expresses the nth derivative of product of two functions in terms of the successive differentiation of the functions and may be thought of as generalization of the theorem concerning the first derivative of the product of two functions.

Statement : If u and v are functions of x possessing nth order derivatives, then

$$(uv)_n = {}^nC_0\, u_n\, v + {}^nC_1\, u_{n-1}\, v_1 + {}^nC_2\, u_{n-2}\, v_2 + \ldots + {}^nC_r\, u_{n-r}\, v_r + \ldots + {}^nC_n\, u\, v_n$$

where u_r denotes the rth order derivative of u and ${}^nC_r$ denotes the number of combinations out of n different things taken r at a time.

Proof. We have

$$(uv)_1 = u_1 v + u v_1 = {}^1C_0\, u_1 v + {}^1C_1\, u\, v_1$$

$\therefore$ theorem is true for $n = 1$.

Assume that the theorem is true for $n = m$, where m is a positive integer.

$$\therefore\ (uv)_m = {}^mC_0\, u_m\, v + {}^mC_1\, u_{m-1}\, v_1 + {}^mC_2\, u_{m-2}\, v_2 + \ldots$$
$$+ {}^mC_{r-1}\, u_{m-r+1}\, v_{r-1} + {}^mC_r\, u_{m-r}\, v_r + \ldots + {}^mC_m\, u\, v_m$$

Differentiating both sides *w.r.t.* x, we get,

$$(uv)_{m+1} = {}^mC_0\, u_{m+1}\, v + {}^mC_0\, u_m\, v_1$$
$$+ {}^mC_1\, u_m\, v_1 + {}^mC_1\, u_{m-1}\, v_2$$
$$+ {}^mC_2\, u_{m-1}\, v_2 + {}^mC_2\, u_{m-2}\, v_3$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$+ {}^mC_{r-1}\, u_{m-r+2}\, v_{r-1} + {}^mC_{r-1}\, u_{m-r+1}\, v_r$$

$$+ {}^mC_r\, u_{m-r+1}\, v_r + {}^mC_r\, u_{m-r}\, v_{r+1}$$

$$+ \ \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$+ {}^mC_m\, u_1\, v_m + {}^mC_m\, u\, v_{m+1}$$

$$\therefore\ (u\,v)_{m+1} = {}^mC_0\, u_{m+1}\, v + ({}^mC_0 + {}^mC_1)\, u_m\, v_1 + ({}^mC_1 + {}^mC_2)\, u_{m-1}\, v_2$$

$$+ \dots + ({}^mC_{r-1} + {}^mC_r)\, u_{m-r+1}\, {}_r + \dots + {}^mC_m\, u\, v_{m+1}$$

But $\quad {}^mC_0 = 1 = {}^{m+1}C_0$

$${}^mC_0 + {}^mC_1 = {}^{m+1}C_1$$

$${}^mC_1 + {}^mC_2 = {}^{m+1}C_2$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$$

$${}^mC_m = 1 = {}^{m+1}C_{m+1}$$

$\therefore$ we have

$$(u\,v)_{m+1} = {}^{m+1}C_0\, u_{m+1}\, v + {}^{m+1}C_1\, u_m\, v_1 + {}^{m+1}C_2\, u_{m-1}\, v_2 + \dots$$

$$+ {}^{m+1}C_r\, u_{m-r+1}\, v_r + \dots + {}^{m+1}C_{m+1}\, u\, v_{m+1}$$

$\therefore$ theorem is true for $n = m + 1$.

$\therefore$ if the theorem is true for $n = m$, then it is also true for $n = m + 1$

But the theorem is true for $n = 1$.

$\therefore$ by the method of induction, theorem is true for all positive integers n.

ILLUSTRATIVE EXAMPLES

Example 1. Find the nth derivative of

(*i*) $e^x \sin x$

(*ii*) $x^2 e^x \cos x$

Sol. (*i*) $y = e^x \sin x$

Let $u = e^x \qquad v = \sin x$

$$u_1 = e^x \qquad v_1 = \sin\left(x + \frac{\pi}{2}\right)$$

$$u_2 = e^x \qquad v_2 = \sin\left(x + 2.\frac{\pi}{2}\right)$$

$$u_n = e^x \qquad v_n = \sin\left(x + n\frac{\pi}{2}\right)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Squaring and cross-multiplying

$$(1-x^2)\,y_1^2 = 4\,(\sin^{-1}x)^2 \quad \Rightarrow \quad (1-x^2)\,y_1^2 = 4\,y \qquad [\because \text{ of } (1)]$$

$$\therefore \quad (1-x^2\,y_1^2 - 4\,y = 0$$

Differentiating *w.r.t.* x, again we get,

$$(1-x^2)\,2\,y_1\,y_2 - 2\,x\,y_1^2 - 4\,y_1 = 0$$

Dividing by $2\,y_1$, we get

$$(1-x^2)\,y_2 - x\,y_1 - 2 = 0 \qquad \ldots(3)$$

Differentiating n times (3) by Leibnitz's rule,

$$1.\,y_{n+2}\,(1-x^2) + \frac{n}{1}\,y_{n+1}\,(-2\,x) + \frac{n\,(n-1)}{2.1}\,y_n\,(-2) - 1\,.\,y_{n+1}\,x - \frac{n}{1}\,y_n\,.\,1 - 0 = 0$$

or $\quad (1-x^2)\,y_{n+2} - (2\,n+1)\,x\,y_{n+1} - n^2\,y_n = 0 \qquad \ldots(4)$

Putting $x = 0$ in (1), (2), (3) and (4), we get,

$$y\,(0) = 0 \qquad \ldots(5)$$

$$y_1\,(0) = 0 \qquad \ldots(6)$$

$$y_2\,(0) = 2 \qquad \ldots(7)$$

$$y_{n+2} = n^2\;y_n\,(0) \qquad \ldots(8)$$

Putting $n = 1, 2, 3, 4 \ldots$ in (8), we get,

$$y_3\,(0) = 1^2\;\;y_1\,(0) = 0 \qquad \ldots(9) \qquad [\because \text{ of } (6)]$$

$$y_4\,(0) = 2^2\;\;y_2\,(0) = 2\,.\,2^2 \qquad \ldots(10) \qquad [\because \text{ of } (7)]$$

$$y_5\,(0) = 3^2\;\;y_3\,(0) = 0 \qquad \ldots(11) \qquad [\because \text{ of } (8)]$$

$$y_6\,(0) = 4^2\;\;y_4\,(0) = 2\,.\,2^2\,.\,4^2 \qquad [\because \text{ of } (10)]$$

and so on .

$$\therefore \text{ In general } \quad y_n\,(0) = \begin{cases} 2.2^2\,.4^2 \ldots (n-2)^2 & \text{when } n \text{ is even and } n \neq 2 \\ 0 & \text{when } n \text{ is odd} \end{cases}$$

Example 5. If $y = \sin\,(m\,\sin^{-1}x)$, prove that

$$(1-x^2)\,y_{n+2} - (2\,n+1)\,x\,y_{n+1} - (n^2 - m^2)\,y_n = 0$$

$$\text{Deduce that } \; y_n\,(0) = \begin{cases} 0 & \text{when } n \text{ is even} \\ m(1^2-m^2)(3^2-m^2)\ldots[(n-2)^2-m^2] & \text{when } n \text{ is odd} \end{cases}$$

Sol. Here $y = \sin\,(m\,\sin^{-1}x)$ $\qquad \ldots(1)$

$$\Rightarrow \qquad y_1 = \cos\,(m\,\sin^{-1}x)\,.\,\frac{m}{\sqrt{1-x^2}} \qquad \ldots(2)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$\Rightarrow \quad \sqrt{1-x^2}\ y_1 = m\cos(m\sin^{-1}x) \quad \Rightarrow \quad (1-x^2)\,y_1^2 = m^2\cos^2(m\sin^{-1}x)$

$\Rightarrow \quad (1-x^2)\,y_1^2 = m^2\,[1-\sin^2(m\sin^{-1}x)]$

$\Rightarrow \quad (1-x^2)\,y_1^2 = m^2\,(1-y^2)$ $\quad [\because \text{ of } (1)]$

Differentiating both sides *w.r.t.* x, we get,

$$(1-x^2)\,.\,2\,y_1\,y_2 + y_1^2\,(-2\,x) = m^2\,(-2\,y\,y_1)$$

Dividing both sides by $2\,y_1$, we get,

$$(1-x^2)\,y_2 - x\,y_1 = -\,m^2\,y$$

$\Rightarrow \quad (1-x^2)\,y_2 - x\,y_1 + m^2\,y = 0$ ...(3)

Differentiating n times by Leibnitz's Theorem, we get,

$$[{}^nC_0\,y_{n+2}\,(1-x^2) + {}^nC_1\,y_{n+1}\,(-2\,x) + {}^nC_2\,y_n\,(-2)]$$
$$-\,[{}^nC_0\,y_{n+1}\,.\,x + {}^nC_1\,y_n\,.\,1] + m^2\,y_n = 0$$

or $(1-x^2)\,y_{n+2} - 2\,n\,x\,y_{n+1} - n\,(n-1)\,y_n - x\,y_{n+1} - n\,y_n + m^2\,y_n = 0$

or $(1-x^2)\,y_{n+2} - (2\,n+1)\,x\,y_{n+1} - (n^2-m^2)\,y_n = 0$ ...(4)

Putting $x = 0$ in (1), (2), (3), (4), we get,

$y\,(0) = 0$ ...(5)

$y_1\,(0) = m$ ...(6)

$y_2\,(0) = 0$ ...(7)

$y_{n+2}\,(0) = (n^2-m^2)\ y_n\,(0)$ ...(8)

Putting $n = 1, 2, 3, 4, \ldots$ in (8), we get,

$y_3\,(0) = (1^2-m^2)\ \ y_1\,(0) = (1^2-m^2)\,.\,m$

$y_4\,(0) = (2^2-m^2)\ \ y_2\,(0) = 0$

$y_5\,(0) = (3^2-m^2)\ \ y_3\,(0) = (3^2-m^2)\,(1^2-m^2)\,.\,m$

$y_6\,(0) = (4^2-m^2)\ \ y_4\,(0) = (4^2-m^2)\,.\,0 = 0$

and so on

$$\therefore \text{ in general, } y_n\,(0) = \begin{cases} 0 & \text{when } n \text{ is even} \\ m(1^2-m^2)(3^2-m^2)\ldots[(n-2)^2-m^2] & \text{when } n \text{ is odd} \end{cases}$$

Example 6. If $y = \log\left[x + \sqrt{x^2+1}\right]$, find $y_n\,(0)$.

Sol. Here $y = \log\left(x + \sqrt{1+x^2}\right)$ ...(1)

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\therefore \quad y_1 = \frac{1}{x+\sqrt{1+x^2}} \cdot \left[1+\frac{2x}{2\sqrt{1+x^2}}\right] \Rightarrow y_1 = \frac{1}{x+\sqrt{1+x^2}} \times \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}$$

$$\therefore \quad y_1 = \frac{1}{\sqrt{1+x^2}} \qquad \ldots(2)$$

$$\Rightarrow \sqrt{1+x^2}\ y_1 = 1 \quad \Rightarrow \quad (1+x^2)\, y_1^2 = 1$$

Differentiating both sides *w.r.t.* x, we get,

$$(1+x^2) \,.\, 2\, y_1 y_2 + y_1^2\,(2x) = 0$$

Dividing both sides by $2\, y_1$, we get,

$$(1+x^2)\, y_2 + x\, y_1 = 0 \qquad \ldots(3)$$

Again differentiating n times, we get,

$$[\,{}^nC_0\, y_{n+2}\,(1+x^2) + {}^nC_1\, y_{n+1}\,(2x) + {}^nC_2\ y_n \,.\, (2)] + [\,{}^nC_0\, y_{n+1} \,.\, x + {}^nC_1\ y_n \,.\, 1] = 0$$

$$\Rightarrow \quad (1+x^2) \,.\, y_{n+2} + 2\, n\, x\, y_{n+1} + n\,(n-1)\, y_n + n\, x\, y_{n+1} + n\, y_n = 0$$

$$\Rightarrow \quad (1+x^2)\, y_{n+2} + (2\,n+1)\, x\, y_{n+1} + n^2\ y_n = 0 \qquad \ldots(4)$$

Putting $x = 0$ in (4), we get,

$$\therefore \quad y_{n+2}\,(0) + 0 + n^2\, y_n\,(0) = 0 \quad \Rightarrow \quad y_{n+2}\,(0) = -\,n^2\, y_n\,(0).$$

Putting $x = 0$ in (1), (2), (3), (4), we get,

$$y\,(0) = 0 \qquad \ldots(5)$$

$$y_1\,(0) = 1 \qquad \ldots(6)$$

$$y_2\,(0) = 0 \qquad \ldots(7)$$

$$y_{n+2}\,(0) = -\,n^2\ y_n\,(0) \qquad \ldots(8)$$

Putting $n = 1, 2, 3, 4, \ldots$ in (8), we get,

$$y_3\,(0) = -1^2 \cdot y_1\,(0) = -1^2 \cdot 1$$

$$y_4\,(0) = -\,2^2 \,.\, y_2(0) = 0$$

$$y_5\,(0) = 3^2 \cdot y_3(0) = (-3^2)\,(-1^2) \cdot 1$$

$$y_6\,(0) = -\,4^2\ y_4\,(0) = 0$$

and so on

$$\therefore \quad \text{in general,} \quad y_n\,(0) = \begin{cases} (-1)^{\frac{n-1}{2}}\, 1.1^2 3^2 \ldots (n-2)^2, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 7. Find the value of the nth derivative of $e^{m\sin^{-1}x}$ at $x = 0$.

Sol. Let $y = e^{m\sin^{-1}x}$ ...(1)

$$\therefore \quad y_1 = e^{m\sin^{-1}x}\frac{m}{\sqrt{1-x^2}} \qquad ...(2)$$

$$\therefore \quad y_1 = y \cdot \frac{m}{\sqrt{1-x^2}}$$

Squaring and cross-multiplying, we get,

$$(1-x^2)\,y_1^2 = m^2 y^2$$

Differentiating *w.r.t.* x again,

$$(1-x^2)2\,y_1 y_2 - 2\,x\,y_1^2 = m^2 \cdot 2\,y\,y_1$$

Dividing by $2\,y_1$, we get

$$(1-x^2)\,y_2 - x\,y_1 = m^2 y$$

$$\therefore \quad (1-x^2)\,y_2 - x\,y_1 - m^2 y = 0 \qquad ...(3)$$

Differentiating n times (3) by Leibnitz's rule, we get,

$$1 \cdot y_{n+2}(1-x^2) + \frac{n}{1}y_{n+1}(-2x) + \frac{n(n-1)}{2.1}y_n \cdot (-2) - 1 \cdot y_{n+1} \cdot x - \frac{n}{1}y_n \cdot 1 - m^2 y_n = 0$$

$$\therefore \quad (1-x^2)\,y_{n+2} - (2n+1)\,x\,y_{n+1} - (m^2+n^2)\,y_n = 0 \qquad ...(4)$$

Putting $x = 0$ in (1), (2), (3) and (4), we get,

$$y(0) = 1 \qquad ...(5)$$

$$y_1(0) = m \qquad ...(6)$$

$$y_2(0) = m^2 \cdot 1 = m^2 \qquad ...(7)$$

$$y_{n+2}(0) = (m^2+n^2)\,y_n(0) \qquad ...(8)$$

Now putting $n = 1, 2, 3, 4 \ldots$ in (8), we get,

$y_3(0) = (m^2+1^2)\;\; y_1(0) = m\,(m^2+1^2)$

$y_4(0) = (m^2+2^2)\;\; y_2(0) = m^2\,(m^2+2^2)$

$y_5(0) = (m^2+3^3)\;\; y_3(0) = m\,(m^2+1^2)\,(m^2+3^2)$

$y_6(0) = (m^2+4^2)\;\; y_4(0) = m^2\,(m^2+2^2)\,(m^2+4^2)$

and so on.

$\therefore$ in general

$$y_n(0) = \begin{cases} m^2(2^2+m^2)(4^2+m^2)\ldots\{(n-2)^2+m^2\} & \text{when } n \text{ is even} \\ m(1^2+m^2)(3^2+m^2)\ldots\{(n-2)^2+m^2\} & \text{when } n \text{ is odd} \end{cases}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 8. If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, prove that

$$(x^2-1)\,y_{n+2} + (2n+1)\,x\,y_{n+1} + (n^2-m^2)\,y_n = 0.$$

Sol. $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x \quad \Rightarrow \quad y^{\frac{1}{m}} + \dfrac{1}{y^{\frac{1}{m}}} = 2x$

$$\Rightarrow \quad y^{\frac{2}{m}} + 1 = 2x\,y^{\frac{1}{m}} \quad \Rightarrow \quad y^{\frac{2}{m}} - 2x\,y^{\frac{1}{m}} + 1 = 0$$

$$\Rightarrow \quad y^{\frac{1}{m}} = \frac{2x \pm \sqrt{4x^2-4}}{2} \quad \Rightarrow \quad y^{\frac{1}{m}} = x \pm \sqrt{x^2-1}$$

$$\Rightarrow \quad y = \left(x+\sqrt{x^2-1}\right)^m, \left(x-\sqrt{x^2-1}\right)^m$$

Let us take $y = \left(x+\sqrt{x^2-1}\right)^m$

$$\therefore \quad y_1 = m\left(x+\sqrt{x^2-1}\right)^{m-1} \times \left[1+\frac{2x}{2\sqrt{x^2-1}}\right]$$

$$= m\left(x+\sqrt{x^2-1}\right)^{m-1} \times \frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}} = \frac{m\left(x+\sqrt{x^2-1}\right)^m}{\sqrt{x^2-1}}$$

$$\therefore \quad y_1 = \frac{my}{\sqrt{x^2-1}} \quad \Rightarrow \quad y_1^2 = \frac{m^2y^2}{x^2-1} \qquad \ldots(1)$$

Again when $y = \left(x-\sqrt{x^2-1}\right)^m$,

$$y_1 = m\left(x-\sqrt{x^2-1}\right)^{m-1} \times \left[1-\frac{2x}{2\sqrt{x^2-1}}\right]$$

$$= m\left(x-\sqrt{x^2-1}\right)^{m-1} \times \left[\frac{\sqrt{x^2-1}-x}{\sqrt{x^2-1}}\right]$$

$$= -m\,\frac{\left(x-\sqrt{x^2-1}\right)^m}{\sqrt{x^2-1}} = -\frac{my}{\sqrt{x^2-1}}$$

$$\Rightarrow \quad y_1^2 = \frac{m^2y^2}{x^2-1} \qquad \ldots(2)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

From (1) and (2), we get,

$$y_1^2 = \frac{m^2 y^2}{x^2 - 1} \quad \Rightarrow \quad (x^2 - 1)\, y_1^2 = m^2 y^2$$

Differentiating both sides *w.r.t.* x, we get,

$$(x^2 - 1) \,.\, 2\, y_1 y_2 + y_1^2\, (2\, x) = m^2 \,.\, 2\, y\, y_1$$

Dividing both sides by $2\, y_1$, we get, $(x^2 - 1)\, y_2 + x\, y_1 = m^2 y$.

Differential both sides n times, we get,

$$[{}^n C_0\, y_{n+2}\, (x^2 - 1) + {}^n C_1\, y_{n+1}\, (2\, x) + {}^n C_2\, y_n \,.\, (2)] + [{}^n C_0\, y_{n+1} \,.\, x + {}^n C_1\, y_n \,.\, 1] = m^2 y_n$$

$$\Rightarrow \quad (x^2 - 1)\, y_{n+2} + 2\, n\, x\, y_{n+1} + n\,(n-1)\, y_n + x\, y_{n+1} + n\, y_n - m^2 y_n = 0$$

$$\Rightarrow \quad (x^2 - 1)\, y_{n+2} + (2\, n + 1)\, x\, y_{n+1} + (n^2 - m^2)\, y_n = 0.$$

Example 9. If $I_n = \dfrac{d^n}{dx^n}\,(x^n \log x)$, prove that $I_n = n\, I_{n-1} + \lfloor n-1$

Hence show that $I_n = \lfloor n \left(\log x + 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n} \right)$.

Sol. $I_n = \dfrac{d^n}{dx^n}\,(x^n \log x)$ ...(1)

$$\therefore \quad I_n = \frac{d^{n-1}}{dx^{n-1}}\left[\frac{d}{dx}(x^n \log x)\right] = \frac{d^{n-1}}{dx^{n-1}}\left[x^n \,.\, \frac{1}{x} + (\log x) \times n\, x^{n-1}\right]$$

$$= \frac{d^{n-1}}{dx^{n-1}}[x^{n-1} + n\, x^{n-1} \log x] = \frac{d^{n-1}}{dx^{n-1}}(x^{n-1}) + \frac{d^{n-1}}{dx^{n-1}}(n\, x^{n-1} \log x)$$

$$= \lfloor n-1 + n \frac{d^{n-1}}{dx^{n-1}}\,(x^{n-1} \log x)$$

$$\therefore \quad I_n = \lfloor n-1 + n\, I_{n-1} \qquad [\because \text{ of } (1)]$$

$$\Rightarrow \quad I_n = n\, I_{n-1} + \lfloor n-1 \qquad \ldots(2)$$

$$\Rightarrow \quad I_n = n\,[(n-1)\, I_{n-2} + \lfloor n-2\,] + \lfloor n-1$$

$$[\because \text{ from } (2),\ I_{n-1} = (n-1)\, I_{n-2} + \lfloor n-2\,]$$

$$\Rightarrow \quad I_n = n\,(n-1)\, I_{n-2} + n \lfloor n-2 + \lfloor n-1$$

$$\Rightarrow \quad I_n = n\,(n-1)\, I_{n-2} + \lfloor n \left[\frac{1}{n-1} + \frac{1}{n}\right]$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Similarly $I_n = n(n-1)(n-2)\, I_{n-3} + \lfloor n \left[\frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}\right]$

and so on.

$$\therefore \quad I_n = n(n-1)(n-2) \,.\, 1 \,.\, I_0 + \lfloor n \left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{n}\right]$$

$$\Rightarrow \quad I_n = \lfloor n \log x + \lfloor n \left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right]$$

$$\Rightarrow \quad \frac{d^n}{dx^n}(x^n \log x) = \lfloor n \left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right].$$

EXERCISE 3 (c)

1. Find the nth derivative of the followings :

   (i) $e^x \log x$ (ii) $x^3 \cos x$ (iii) $x^2 \log x$

   (iv) $e^x \sin x \sin 2x$ (v) $e^{ax} . x$ (vi) $x^n e^x$

   (vii) $x^2 a^x$ (viii) $x \tan^{-1} x$ (ix) $x^2 e^{ax} \sin bx$

   (x) $e^{ax}[b^2 x^2 - 2nax + n(n+1)]$.

2. (a) If $y = x^2 e^x$, show that

$$y_n = \frac{1}{2} n(n-1) y_2 - n(n-2) y_1 + \frac{1}{2}(n-1)(n-2) y$$

   (b) If $y = x^2 e^x$, then prove that $\frac{d^8 y}{dx^8} = 28 \frac{d^2 y}{dx^2} - 48 \frac{dy}{dx} + 21 y$.

3. If $y = x^2 \sin x$, prove that

$$y_n = (x^2 - n^2 + n) \sin\left(x + \frac{n\pi}{2}\right) - 2nx \cos\left(x + \frac{n\pi}{2}\right).$$

4. If $f(x) = \tan x$, prove that

$$f^n(0) - {}^nC_2 f^{n-2}(0) + {}^nC_4 f^{n-4}(0) - \ldots = \sin \frac{n\pi}{2}.$$

5. If $y = x^n \log x$, prove that $y_{n+1} = \frac{\lfloor n}{x}$.

6. If $x = \sin\left(\frac{\log y}{a}\right)$, prove that

$$(1 - x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2 + a^2) y_n = 0.$$

7. If $y = \tan^{-1} x$, prove that $(1 + x^2) y_{n+1} + 2nx y_n + n(n-1) y_{n-1} = 0$.

   Also prove that $y_n(0)$ is zero or $(-1)^{\frac{n-1}{2}} \lfloor n-1$ according as n is even or odd.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2023-Grant

Gandhi Memorial College Of Education Bantalab Jammu

8. If $y = \cos(m \sin^{-1} x)$, find $y_n(0)$.

9. If $y = \log\left(x + \sqrt{1+x^2}\right)$ prove that $(1+x^2)\, y_2 + x\, y_1 = 0$. Differentiate this n times and deduce that $y_{n+2}(0) = -n^2\, y_n(0)$.

10. If $y = \left[\log\left(x + \sqrt{x^2+1}\right)\right]^2$ prove that

$$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0.$$

Hence find the value of $y_n(0)$.

11. If $y = \left(x + \sqrt{1+x^2}\right)^m$ find $y_n(0)$.

12. If $y = e^{\tan^{-1} x}$, show that $(1+x^2)\, y_2 + (2x-1)\, y_1 = 0$ and $(1+x^2)\, y_{n+2} + \{2(n+1)x - 1\}\, y_{n+1} + n(n+1)\, y_n = 0$.

OR

If $x = \tan(\log y)$, prove that

$(1+x^2)\, y_{n+2} + \{2(n+1)x - 1\}\, y_{n+1} + n(n+1)\, y_n = 0$.

13. If $y = e^{m\cos^{-1} x}$, show that $(1-x^2)\, y_{n+2} - (2n+1)\, x\, y_{n+1} - (n^2+m^2)\, y_n = 0$. Also find $y_n(0)$.

14. If $y = e^{a\sin^{-1} x}$ prove that $(1-x^2)\, y_{n+2} - (2n+1)\, x\, y_{n+1} - (n^2+a^2)\, y_n = 0$

and $\underset{x \to 0}{\text{Lt}} \dfrac{y_{n+2}}{y_n} = n^2 + a^2$.

15. If $y = (x^2-1)^n$, prove that $(x^2-1)\, y_{n+2} + 2x\, y_{n+1} - n(n+1)\, y_n = 0$.

16. If $\cos^{-1}\left(\dfrac{y}{b}\right) = \log\left(\dfrac{x}{n}\right)^n$, then prove that

$x^2 y_{n+2} + (2n+1)\, x\, y_{n+1} + 2n^2 y_n = 0$.

ANSWERS

1. (*i*) $e^x \left[\log x + {}^nC_1 \dfrac{1}{x} - {}^nC_2 \dfrac{1}{x^2} + \ldots + (-1)^{n-1}\, {}^nC_n \, \underline{|n-1} \, \dfrac{1}{x^n}\right]$

(*ii*) $x^3 \cos\left(x + \dfrac{n\pi}{2}\right) + 3nx^2 \cos\left(x + \overline{n-1}\,\dfrac{\pi}{2}\right) + 3n(n-1)\, x \cos\left(x + \overline{n-2}\,\dfrac{\pi}{2}\right)$

$+ n(n-1)(n-2) \cos\left(x + \overline{n-3}\,\dfrac{\pi}{2}\right)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

(iii) $\dfrac{(-1)^{n-1}\lfloor n}{x^{n-2}}\left[\dfrac{1}{n}-\dfrac{2}{n-1}+\dfrac{1}{n-2}\right]$ if $n>2$

(iv) $\dfrac{1}{2}e^{x}\left[\cos x+{}^{n}C_{1}\cos\left(x+\dfrac{\pi}{2}\right)+{}^{n}C_{2}\cos\left(x+2\dfrac{\pi}{2}\right)\right.$

$\left.+\ldots\ldots+{}^{n}C_{n}\cos\left(x+\dfrac{n\pi}{2}\right)\right]$

(v) $e^{ax}(a^{n}x+n\,.\,a^{n-1})$

(vi) $e^{x}\left[x^{n}+\dfrac{n^{2}}{\lfloor 1}x^{n-1}+\dfrac{n^{2}(n-1)^{2}}{\lfloor 2}x^{n-2}+\ldots+\dfrac{n^{2}(n-1)^{2}(n-2)^{2}\ldots 1^{2}}{\lfloor n}\right]$

(vii) $a^{x}[x^{2}(\log a)^{n}+2nx(\log a)^{n-1}+n(n-1)(\log a)^{n-2}]$

(viii) $(-1)^{n-1}\lfloor n-2\ \sin^{n-1}\theta\cos\theta\cos n\theta\,[n\tan\theta-\tan n\theta]$ where $x=\cot\theta$

(ix) $x^{2}(a^{2}+b^{2})^{\frac{n}{2}}e^{ax}\sin\left(bx+n\tan^{-1}\dfrac{b}{a}\right)$

$+2nx(a^{2}+b^{2})^{\frac{n-1}{2}}e^{ax}\sin\left(bx+\overline{n-1}\tan^{-1}\dfrac{b}{a}\right)$

$+n(n-1)(a^{2}+b^{2})^{\frac{n-2}{2}}e^{ax}\sin\left(ax+\overline{n-2}\tan^{-1}\dfrac{b}{a}\right)$

(x) $a^{n+2}e^{ax}\,.\,x^{2}$

7. $y_n(0)=\begin{cases} m^{2}(m^{2}-2^{2})(m^{2}-4^{2})\ldots[m^{2}-(n-2)^{2}] & \text{when } n \text{ is even}\\ m\ (m^{2}-1^{2})(m^{2}-3^{2})\ldots[m^{2}-(n-2)^{2}] & \text{when } n \text{ is odd}\end{cases}$

8. $y_n(0)=\begin{cases} 0, & \text{when } n \text{ is odd}\\ [(n-2)^{2}-m^{2}]\ldots(4^{2}-m^{2})(2^{2}-m^{2})(-m^{2}), & \text{when } n \text{ is even}\end{cases}$

11. $y_n(0)=\begin{cases} m^{2}(m^{2}-2^{2})(m^{2}-4^{2})\ldots[m^{2}-(n-2)^{2}] & \text{when } n \text{ is even}\\ m\ (m^{2}-1^{2})(m^{2}-3^{2})\ldots[m^{2}-(n-2)^{2}] & \text{when } n \text{ is odd}\end{cases}$

15. $y_n(0)=\begin{cases} (-1)^{\frac{n}{2}-1}\,2.2^{2}.4^{2}\ldots(n-2)^{2} & \text{when } n \text{ is even}\\ 0 & \text{when } n \text{ is odd}\end{cases}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

4

INDETERMINATE FORMS

Art-1. Indeterminate Forms

We have earlier pointed out in the beginning of this chapter that while evaluating limits, we may come across the situations $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \,.\, \infty, 0^0, \infty^0, 1^\infty$, which are called indeterminate forms. For evaluating the form $\frac{0}{0}$, we use L' Hospital Rule which is given below.

Art-2. L'Hospital Rule

Statement. If f, g are two functions such that

(*i*) $\underset{x \to a}{\text{Lt}} f(x) = \underset{x \to a}{\text{Lt}} g(x) = 0$

(*ii*) $f'(x), g'(x)$ both exist and $g'(x) \neq 0 \; \forall \, x \in (a - \delta, a + \delta), \; \delta > 0$ except possibly at $x = a$.

(*iii*) $\underset{x \to a}{\text{Lt}} \frac{f'(x)}{g'(x)}$ exists (finitely or infinitely), then $\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{g'(x)}$.

Proof. Let us define two functions F and G such that

$$\text{F}(x) = \begin{cases} f(x), \; \forall \; x \in (a - \delta, a + \delta), \; x \neq a \\ 0 \;, \quad x = a \end{cases}$$

$$\text{G}(x) = \begin{cases} g(x), \; \forall \; x \in (a - \delta, a + \delta), \; x \neq a \\ 0 \;, \quad x = a \end{cases}$$

Let x be any real number such that $a < x < a + \delta$ then

1. F, G are both continuous in $[a, x]$

$$\left[\because \underset{x \to a}{\text{Lt}} \text{F}(x) = \underset{x \to a}{\text{Lt}} f(x) = 0 = \text{F}(a), \text{ etc.} \right]$$

2. F, G are both derivable in (a, x)
3. G' is not zero anywhere in (a, x).

$\therefore$ F and G satisfy all the conditions of Cauchy's Mean Value theorem.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$\therefore$ there exists atleast one real number $c \in (a, x)$ such that

$$\frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F'(c)}{G'(c)}, \text{ where } a < c < x$$

or $$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \quad \ldots(1)$$

Now when $x \to a+$, $c \to a+$

$\therefore$ from (1), we get

$$\underset{x \to a+}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{c \to a+}{\text{Lt}} \frac{f'(c)}{g'(c)} = \underset{x \to a+}{\text{Lt}} \frac{f'(x)}{g'(x)}$$

$$\therefore \quad \underset{x \to a+}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x \to a+}{\text{Lt}} \frac{f'(x)}{g'(x)} \quad \ldots(2)$$

Similarly $$\underset{x \to a-}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x \to a-}{\text{Lt}} \frac{f'(x)}{g'(x)} \quad \ldots(3)$$

From (2) and (3), we get

$$\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{g'(x)}$$

Note 1. If $\underset{x \to a}{\text{Lt}} \frac{f'(x)}{g'(x)}$ again takes the form $\frac{0}{0}$, we repeat the process or use Taylor's Theorem.

Note 2. L'Hospital Rule when $x \to \infty$

This rule holds even when $x \to \infty$. Its statement is :

If f, g are two functions such that

(*i*) $\underset{x \to \infty}{\text{Lt}} f(x) = \underset{x \to \infty}{\text{Lt}} g(x) = 0$

(*ii*) $f'(x)$, $g'(x)$ both exist and $g'(x) \neq 0 \ \forall \ x > 0$ except possibly at ∞

(*iii*) $\underset{x \to \infty}{\text{Lt}} \frac{f'(x)}{g'(x)}$ exists, then $\underset{x \to \infty}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x \to \infty}{\text{Lt}} \frac{f'(x)}{g'(x)}$

Same rule holds when $x \to -\infty$.

Note 3. Standard results $\underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = 1$, $\underset{x \to 0}{\text{Lt}} \frac{\tan x}{x} = 1$, $\underset{x \to 0}{\text{Lt}} (1+x)^{\frac{1}{x}} = e$ etc. should be used before applying L' Hospital Rule.

Note 4. Sometimes, we shall be using standard expansions in evaluating limits of the form $\frac{0}{0}$. The use of expansions reduces the labour of differentiating time and again. So readers are advised to remember the following expansions. We shall be using these in some of the illustrative examples.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Some Standard Expansions.

(*i*) $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots$ (*ii*) $e^{-x} = 1 - x + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} + \ldots$

(*iii*) $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \ldots$ (*iv*) $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \ldots$

(*v*) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$ (*vi*) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{15} - \ldots$

(*vii*) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

(*viii*) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$

(*ix*) $(1+x)^{\frac{1}{x}} = e - \frac{ex}{2} + \frac{11ex^2}{24} + \ldots$ (near $x = 0$)

Art-3. Indeterminate Form $\frac{0}{0}$

We give some examples to explain the method.

ILLUSTRATIVE EXAMPLES

Example 1. Evaluate the limit : $\underset{x \to 0}{Lt} \left(\frac{e^x - e^{-x}}{\sin x} \right)$.

Sol. $\underset{x \to 0}{Lt} \left(\frac{e^x - e^{-x}}{\sin x} \right)$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \underset{x \to 0}{Lt} \left(\frac{e^x + e^{-x}}{\cos x} \right) = \frac{e^0 + e^0}{\cos 0} = \frac{1+1}{1} = 2$$

Example 2. Evaluate $\underset{x \to 0}{\text{Lt}} \frac{1 - \cos^2 x}{\sin x^2}$.

Sol. $\underset{x \to 0}{\text{Lt}} \frac{1 - \cos^2 x}{\sin x^2}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{2 \sin x \cos x}{2x \cos x^2}$$ [∵ of L' Hospital Rule]

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$= \underset{x \to 0}{\text{Lt}} \frac{\sin x \cos x}{x \cos x^2} = \underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} \cdot \underset{x \to 0}{\text{Lt}} \frac{\cos x}{\cos x^2} = (1) \cdot \frac{1}{1}$$

$$\left[\because \underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = 1, \underset{x \to 0}{\text{Lt}} \cos x = 1\right]$$

$$= 1.$$

Example 3. Evaluate $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$.

Sol. $$\underset{x \to 0}{\text{Lt}} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x} = \underset{x \to 0}{\text{Lt}} \frac{e^x - e^{-x} - 2\log(1+x)}{x^2} \cdot \frac{x}{\sin x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x - e^{-x} - 2\log(1+x)}{x^2} \cdot 1 \qquad \left(\because \underset{x \to 0}{\text{Lt}} \frac{x}{\sin x} = 1\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x - e^{-x} - 2\log(1+x)}{x^2} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x + e^{-x} - \dfrac{2}{1+x}}{2x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e^x - e^{-x} + \dfrac{2}{(1+x)^2}}{2} = \frac{1 - 1 + \dfrac{2}{1}}{2} = \frac{2}{2} = 1.$$

Example 4. Evaluate $\underset{x \to 0}{\text{Lt}} \dfrac{1 + \sin x - \cos x + \log(1-x)}{x \tan^2 x}$.

Sol. $$\underset{x \to 0}{\text{Lt}} \frac{1 + \sin x - \cos x + \log(1-x)}{x \tan^2 x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{1 + \sin x - \cos x + \log(1-x)}{x^3} \left(\frac{x}{\tan x}\right)^2$$

$$= \underset{x \to 0}{\text{Lt}} \frac{1 + \sin x - \cos x + \log(1-x)}{x^3} \times (1)^2 \qquad \left[\because \underset{x \to 0}{\text{Lt}} \frac{x}{\tan x} = 1\right]$$

$$= \underset{x \to 0}{\text{Lt}} \frac{1 + \sin x - \cos x + \log(1-x)}{x^3} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\cos x + \sin x - \dfrac{1}{1-x}}{3x^2} \qquad \left(\frac{0}{0} \text{ form}\right)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$= \underset{x \to 0}{\text{Lt}} \frac{-\sin x + \cos x - \dfrac{1}{(1-x)^2}}{6x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-\cos x - \sin x - \dfrac{2}{(1-x)^3}}{6} = \frac{-1-0-2}{6} = -\frac{3}{6} = -\frac{1}{2}.$$

Example 5. Prove that $\underset{x \to 0}{\text{Lt}} \dfrac{x e^x - \log(1+x)}{x^2} = \dfrac{3}{2}$

Sol. $\underset{x \to 0}{\text{Lt}} \dfrac{x e^x - \log(1+x)}{x^2}$ $\left[\text{It is of the form } \dfrac{0}{0} \text{ but let us use expansion}\right]$

$$= \underset{x \to 0}{\text{Lt}} \frac{x\left(1 + x + \dfrac{x^2}{\lfloor 2} + \ldots\right) - \left(x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \ldots\right)}{x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{(x-x) + \left(x^2 + \dfrac{x^2}{2}\right) + \left(\dfrac{x^3}{2} - \dfrac{x^3}{3}\right) + \ldots\ldots}{x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\dfrac{3x^2}{2} + \dfrac{x^3}{6} + \ldots}{x^2} = \underset{x \to 0}{\text{Lt}} \frac{x^2\left(\dfrac{3}{2} + \dfrac{x}{6} + \ldots\right)}{x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x^2\left(\dfrac{3}{2} + \text{terms containing } x \text{ and its higher powers}\right)}{x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \left(\frac{3}{2} + \text{terms containing } x \text{ and its higher powers}\right)$$

$$= \frac{3}{2} + 0 = \frac{3}{2}.$$

Example 6. Evaluate $\underset{x \to 0}{\text{Lt}} \dfrac{(1+\sin x)^{\frac{1}{3}} - (1-\sin x)^{\frac{1}{3}}}{x}$

Sol. $\underset{x \to 0}{\text{Lt}} \dfrac{(1+\sin x)^{\frac{1}{3}} - (1-\sin x)^{\frac{1}{3}}}{x}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$= \underset{x\to 0}{\text{Lt}} \frac{\left[1+\frac{1}{3}\sin x+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{\lfloor 2}\sin^2 x+\ldots\right]-\left[1-\frac{1}{3}\sin x+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{\lfloor 2}\sin^2 x+\ldots\right]}{x}$$

$$= \frac{\frac{2}{3}\sin x-\frac{10}{81}\sin^3 x+\ldots\ldots}{x} = \frac{2}{3}\frac{\sin x}{x}-\frac{10}{81}\frac{\sin x}{x}.\sin^2 x+\ldots\ldots$$

$$= \frac{2}{3}.1-\frac{10}{81}.1.0+0 = \frac{2}{3}-0 = \frac{2}{3}$$

Example 7. Prove that $\underset{x\to 1}{\text{Lt}} \frac{x^x-x}{x-1-\log x} = 2$

Sol. $\underset{x\to 1}{\text{Lt}} \frac{x^x-x}{x-1-\log x}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \underset{x\to 1}{\text{Lt}} \frac{x^x(1+\log x)-1}{1-\frac{1}{x}}$$

$$\left[\text{Put } y=x^x,\ \therefore\ \log y = x\log x \Rightarrow \frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x}+\log x.1 \Rightarrow \frac{dy}{dx} = x^x(1+\log x)\right]$$

$$= \underset{x\to 1}{\text{Lt}} \frac{x^x\times\frac{1}{x}+(1+\log x).x^x(1+\log x)}{\frac{1}{x^2}}$$

$$= \frac{1\times 1+(1+0)\times 1(1+0)}{1} = \frac{1+1}{1} = 2.$$

Example 8. Evaluate the following :

(*i*) $\underset{x\to 0}{\text{Lt}} \frac{(1+x)^{\frac{1}{x}}-e}{x}$ (*ii*) $\underset{x\to 0}{\text{Lt}} \frac{(1+x)^{\frac{1}{x}}-e+\frac{1}{2}ex}{x^2}$

(*iii*) $\underset{x\to 0}{\text{Lt}} \frac{(1+x)^{\frac{1}{x}}-e+\frac{1}{2}ex-\frac{11}{24}ex^2}{x^3}$

Sol. Let $y = (1+x)^{\frac{1}{x}}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\therefore \quad \log y = \log (1+x)^{\frac{1}{x}} = \frac{1}{x}\log(1+x)$$

$$= \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots$$

$$\therefore \quad \log y = 1 + t, \text{ where } t = -\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots$$

$$\therefore \quad y = e^{1+t} = e \,.\, e^t = e\left[1 + \frac{t}{\lfloor 1} + \frac{t^2}{\lfloor 2} + \frac{t^3}{\lfloor 3} + \ldots\right]$$

$$\therefore \quad y = e \,.\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots\right) + \frac{1}{2}\left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots\right)^2 + \frac{1}{6}\left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots\right)^3 + \ldots\right]$$

$$= e\left[1 - \frac{1}{2}x + \left(\frac{1}{3} + \frac{1}{8}\right)x^2 + \left(-\frac{1}{4} - \frac{1}{6} - \frac{1}{48}\right)x^3 + \ldots\right]$$

$$= e\left[1 - \frac{1}{2}x + \frac{11}{24}x^2 - \frac{7}{16}x^3 + \ldots\right].$$

(*i*) $$\underset{x \to 0}{\text{Lt}} \frac{(1+x)^{\frac{1}{x}} - e}{x} = \underset{x \to 0}{\text{Lt}} \frac{e\left(1 - \frac{1}{2}x + \ldots\right) - e}{x}$$

$$= \underset{x \to 0}{\text{Lt}} \left[\frac{-\frac{1}{2}ex + \text{terms contains } x^2 \text{ and higher power of } x}{x}\right]$$

$$= \underset{x \to 0}{\text{Lt}} \left[-\frac{1}{2}e + \text{terms containing } x \text{ and its higher power}\right]$$

$$= -\frac{1}{2}e + 0 = -\frac{1}{2}e.$$

(*ii*) $$\underset{x \to 0}{\text{Lt}} \frac{(1+x)^{\frac{1}{x}} - e + \frac{1}{2}ex}{x^2} = \underset{x \to 0}{\text{Lt}} \frac{e\left(1 - \frac{1}{2}x + \frac{11}{24}x^2 - \ldots\right) - e + \frac{1}{2}ex}{x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\frac{11}{24}ex^2 + \ldots}{x^2}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$= \underset{x \to 0}{\text{Lt}} \left[\frac{11}{24} e + \text{terms containing } x \text{ and its higher powers} \right]$$

$$= \frac{11}{24} + 0 = \frac{11}{24}.$$

(*iii*) $$\underset{x \to 0}{\text{Lt}} \frac{(1+x)^{\frac{1}{x}} - e + \frac{1}{2} e x - \frac{11}{24} e x^2}{x^3}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e\left(1 - \frac{1}{2}x + \frac{11}{24}x^2 - \frac{7}{16}x^3 + \ldots\right) - e + \frac{1}{2} e x - \frac{11}{24} e x^2}{x^3}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-\frac{7}{16} e x^3 + \ldots}{x^3}$$

$$= \underset{x \to 0}{\text{Lt}} \left[-\frac{7}{16} e + \text{terms containing } x \text{ and its higher powers} \right]$$

$$= -\frac{7}{16} e + 0 = -\frac{7}{16} e.$$

Example 9. Find the values of a and b so that $\underset{x \to 0}{\text{Lt}} \dfrac{x(1 + a\cos x) - b\sin x}{x^3}$ exists and equals 1.

Sol. Let $$L = \underset{x \to 0}{\text{Lt}} \frac{x(1 + a\cos x) - b\sin x}{x^3} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x(-a\sin x) + (1 + a\cos x).1 - b\cos x}{3x^2} \qquad [\because \text{ of L' Hospital's Rule}]$$

$$\therefore \quad L = \underset{x \to 0}{\text{Lt}} \frac{-ax\sin x + 1 + a\cos x - b\cos x}{3x^2} \qquad \ldots(1)$$

Now denominator of R.H.S. of (1) is zero when $x = 0$. Therefore, in order that given limit may exist finitely, it is necessary that numerator of R.H.S. of (1) is also zero at $x = 0$.

$$\therefore \quad -a(0)\sin 0 + 1 + a\cos 0 - b\cos 0 = 0$$

$$\therefore \quad 1 + a - b = 0 \qquad \ldots(2)$$

Assume that $1 + a - b = 0$.

$$\therefore \quad L = \underset{x \to 0}{\text{Lt}} \frac{-ax\sin x + 1 + a\cos x - b\cos x}{3x^2} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-a(x\cos x + \sin x.1) - a\sin x + b\sin x}{6x}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$= \underset{x \to 0}{\text{Lt}} \frac{-a x \cos x - 2 a \sin x + b \sin x}{6x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-a(1.\cos x - x \sin x) - 2 a \cos x + b \cos x}{6}$$

$$= \frac{-a(1.1-0) - 2a(1) + b(1)}{6} = \frac{-a - 2a + b}{6} = \frac{b - 3a}{6}$$

Now L = 1 (given)

$$\therefore \quad \frac{b - 3a}{6} = 1$$

$$\Rightarrow \quad b - 3a = 6 \qquad \Rightarrow \qquad b = 6 + 3a \qquad \ldots(3)$$

Substituting $b = 6 + 3a$ in (2), we get,

$$1 + a - 6 - 3a = 0 \qquad \text{or} \qquad -2a = 5$$

$$\therefore \qquad a = -\frac{5}{2}$$

$$\therefore \text{ from (3), } \quad b = 6 - \frac{15}{2} = -\frac{3}{2}$$

$$\therefore \text{ we have } \quad a = -\frac{5}{2}, \; b = -\frac{3}{2}.$$

Example 10. Find a, b and c so that $\underset{x \to 0}{\text{Lt}} \dfrac{a \sin x - b x + c x^2 + x^3}{2 x^2 \log(1+x) - 2 x^3 + x^4}$ is finite. Also determine the limit.

Sol. Let $\text{L} = \underset{x \to 0}{\text{Lt}} \dfrac{a \sin x - b x + c x^2 + x^3}{2 x^2 \log(1+x) - 2 x^3 + x^4} \qquad \left(\dfrac{0}{0} \text{ form}\right)$

$$\therefore \quad \text{L} = \underset{x \to 0}{\text{Lt}} \frac{a \cos x - b + 2 c x + 3 x^2}{\dfrac{2x^2}{1+x} + 4 x \log(1+x) - 6 x^2 + 4 x^3} \qquad \ldots(1)$$

[∵ of L' Hospital's Rule]

Now denominator of R.H.S. of (1) is zero when $x = 0$. Therefore, in order that given limit may exist finitely, it is necessary that numerator of R.H.S. of (1) is also zero at $x = 0$.

$$\therefore \quad a \cos 0 - b + 0 + 0 = 0$$

$$\therefore \quad a - b = 0 \qquad \ldots(2)$$

Assume that $a - b = 0$

$$\therefore \quad \text{L} = \underset{x \to 0}{\text{Lt}} \frac{a \cos x - b + 2 c x + 3 x^2}{\dfrac{2 x^2}{1+x} + 4 x \log(1+x) - 6 x^2 + 4 x^3} \qquad \left(\frac{0}{0} \text{ form}\right)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\therefore \quad L = \underset{x \to 0}{\text{Lt}} \frac{-a \sin x + 2c + 6x}{\dfrac{4x + 2x^2}{(1+x)^2} + \dfrac{4x}{1+x} + 4 \log(1+x) - 12x + 12x^2} \quad \ldots(3)$$

$$\left[\because \frac{d}{dx}\left(\frac{2x^2}{1+x}\right) = \frac{(1+x).4x - 2x^2.1}{(1+x^2)} = \frac{4x+x^2}{(1+x)^2} \right]$$

Now denominator of R.H.S. of (3) is zero when $x = 0$. Therefore, in order that given limit may exist finitely, it is necessary that numerator of R.H.S. of (3) is also zero at $x = 0$.

$\therefore \quad 0 + 2c + 0 = 0 \Rightarrow 2c = 0 \Rightarrow c = 0$

Assume that $c = 0$

$$\therefore \quad L = \underset{x \to 0}{\text{Lt}} \frac{-a \sin x + 6x}{\dfrac{4x + 2x^2}{(1+x)^2} + \dfrac{4x}{1+x} + 4 \log(1+x) - 12x + 12x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-a \cos x + 6}{\left[\dfrac{(1+x)^2 (4+4x) - (4x+2x^2)(2)(1+x)}{(1+x)^4}\right] + \dfrac{(1+x)4 - 4x}{(1+x)^2} + \dfrac{4}{1+x} - 12 + 24x}$$

$$\therefore \quad L = \underset{x \to 0}{\text{Lt}} \frac{-a \cos x + 6}{\dfrac{4}{(1+x)^3} + \dfrac{4}{(1+x)^2} + \dfrac{4}{1+x} - 12 + 24x} \quad \ldots(4)$$

Now denominator of R.H.S. of (4) is zero when $x = 0$. Therefore, in order that given limit may exist finitely, it is necessary that numerator of R.H.S. of (4) is also zero at $x = 0$.

$\therefore \quad -a \cos 0 + 6 = 0 \Rightarrow -a + 6 = 0$

$\therefore \quad a = 6$

$\therefore$ from (2), $b = 6$

$\therefore$ we have $a = 6, b = 6, c = 0$

$$\therefore \quad L = \underset{x \to 0}{\text{Lt}} \frac{6 \sin x}{\dfrac{-12}{(1+x)^4} - \dfrac{8}{(1+x)^3} - \dfrac{4}{(1+x)^2} - 0 + 24} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{6 \cos x}{\dfrac{48}{(1+x)^5} + \dfrac{24}{(1+x)^4} + \dfrac{8}{1+x^3} + 0}$$

$$= \frac{6}{48 + 24 + 8} = \frac{6}{80} = \frac{3}{40}$$

$\therefore$ we have $a = 6, b = 6, c = 0$

and required limit $= \dfrac{3}{40}$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 11. Show that $\underset{x \to 0}{\text{Lt}} \dfrac{x^2 \sin \frac{1}{x}}{\tan x}$ exists, but cannot be evaluated by L' Hospital's Rule. What is the limit ?

Sol. Let $f(x) = x^2 \sin \dfrac{1}{x}$, $g(x) = \tan x$

$$\text{Now} \quad \underset{x \to 0}{\text{Lt}} \frac{f'(x)}{g'(x)} = \underset{x \to 0}{\text{Lt}} \frac{\frac{d}{dx}\left(x^2 \sin \frac{1}{x}\right)}{\frac{d}{dx}(\tan x)} = \underset{x \to 0}{\text{Lt}} \frac{2x \sin \frac{1}{x} - \cos \frac{1}{x}}{\sec^2 x}$$

which does not exist as $\underset{x \to 0}{\text{Lt}} \cos \dfrac{1}{x}$ does not exist.

$\therefore$ one condition of L' Hospital's Rule is not satisfied and consequently L' Hospital's Rule is not applicable.

$$\text{However} \quad \underset{x \to 0}{\text{Lt}} \frac{x^2 \sin \frac{1}{x}}{\tan x} = \underset{x \to 0}{\text{Lt}} \left(\frac{x}{\tan x}\right).\left(x \sin \frac{1}{x}\right) = \underset{x \to 0}{\text{Lt}} \frac{x}{\tan x} . \underset{x \to 0}{\text{Lt}} x \sin \frac{1}{x}$$

$$= (1)\,(0)$$

$$\left[\begin{array}{l} \because \quad \underset{x \to 0}{\text{Lt}}\, x = 0 \text{ and } \sin \frac{1}{x} \text{ is bounded in deleted } nbd. \text{ of } a \\ \therefore \quad \underset{x \to 0}{\text{Lt}}\, x \sin \frac{1}{x} = 0 \\ \text{Also } \underset{x \to 0}{\text{Lt}} \frac{x}{\tan x} = 1 \end{array}\right]$$

$$= 0$$

$$\therefore \quad \underset{x \to 0}{\text{Lt}} \frac{x^2 \sin \frac{1}{x}}{\tan x} = 0$$

$\therefore \quad \underset{x \to 0}{\text{Lt}} \dfrac{x^2 \sin \frac{1}{x}}{\tan x}$ exists and equals 0.

Example 12. Evaluate $\underset{x \to \infty}{\text{Lt}} \dfrac{\sin^{-1} \frac{1}{x}}{\tan \frac{1}{x}}$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Education Bhatinda

Sol. $\underset{x \to \infty}{\text{Lt}} \dfrac{\sin^{-1}\dfrac{1}{x}}{\tan\dfrac{1}{x}}$ [Put $x = \dfrac{1}{h}$ $(h > 0)$ so that $h \to 0$ as $x \to \infty$]

$$= \underset{h \to 0}{\text{Lt}} \frac{\sin^{-1} h}{\tan h} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{h \to 0}{\text{Lt}} \frac{\dfrac{1}{\sqrt{1-h^2}}}{\sec^2 h} = \frac{\dfrac{1}{\sqrt{1-0}}}{\sec^2 0} = \frac{1}{1} = 1.$$

Example 13. What is the fallacy in

$$\underset{x \to 0}{\text{Lt}} \frac{x^3 + x^2 - 2x}{x^2 + x} = \underset{x \to 0}{\text{Lt}} \frac{3x^2 + 2x - 2}{2x + 1} = \underset{x \to 0}{\text{Lt}} \frac{6x + 2}{2} = \frac{0+2}{2} = 1.$$

Sol. $\underset{x \to 0}{\text{Lt}} \dfrac{x^3 + x^2 - 2x}{x^2 + x}$ $\left(\dfrac{0}{0} \text{ form, } \therefore \text{ L'Hospital's rule is applicable}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{3x^2 + 2x - 2}{2x + 1}$$

$\left(\text{It is not of the } \dfrac{0}{0} \text{ form, so, L'Hospital's rule applied is wrong}\right)$

$$= \frac{0 + 0 - 2}{0 + 1} = -\frac{2}{1} = -2.$$

EXERCISE 4 (a)

1. Evaluate the following limits :

(*i*) $\underset{x \to 0}{\text{Lt}} \dfrac{\sin ax}{\sin bx}$ (*ii*) $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - 1}{x}$

(*iii*) $\underset{x \to 0}{\text{Lt}} \dfrac{a^x - 1}{b^x - 1}$ (*iv*) $\underset{x \to 0}{\text{Lt}} \dfrac{(1+x)^n - 1}{x}$

(*v*) $\underset{x \to 1}{\text{Lt}} \dfrac{\log x}{x - 1}$ (*vi*) $\underset{x \to 0}{\text{Lt}} \dfrac{x - \tan x}{x - \sin x}$

(*vii*) $\underset{x \to 0}{\text{Lt}} \dfrac{1 - \cos x}{x^2}$ (*viii*) $\underset{x \to 0}{\text{Lt}} \dfrac{x - \sin x}{x^3}$

(*ix*) $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$ (*x*) $\underset{x \to 0}{\text{Lt}} \dfrac{\log(1+x) - x}{1 - \cos x}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

(*xi*) $\underset{x \to 0}{\text{Lt}} \log(1-x)\cot\frac{\pi x}{2}$

(*xii*) $\underset{x \to 0}{\text{Lt}} \frac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}$

(*xiii*) $\underset{x \to 0}{\text{Lt}} \frac{e^x \sin x - x - x^2}{x^3}$

(*xiv*) $\underset{x \to 0}{\text{Lt}} \frac{e^x - e^{\sin x}}{x - \sin x}$

(*xv*) $\underset{x \to 0}{\text{Lt}} \frac{\log(1-x^2)}{\log\cos x}$

(*xvi*) $\underset{x \to 0}{\text{Lt}} \frac{x - \sin x}{(x\sin x)^{\frac{3}{2}}}$

(*xvii*) $\underset{x \to 0}{\text{Lt}} \frac{x - \tan^{-1} x}{x - \sin x}$

(*xviii*) $\underset{x \to 0}{\text{Lt}} \frac{x\cos x - \log(1+x)}{x^2}$

(*xix*) $\underset{x \to a}{\text{Lt}} \frac{a^x - x^a}{a^a - x^x}$

(*xx*) $\underset{x \to 1}{\text{Lt}} \frac{x^x - x}{1 - x + \log x}$

2. Evaluate the following limits :

(*i*) $\underset{x \to b}{\text{Lt}} \frac{x^b - b^x}{x^x - b^b}$

(*ii*) $\underset{x \to 0}{\text{Lt}} \frac{e^x - 2\cos x + e^{-x}}{x\sin x}$

(*iii*) $\underset{x \to 0}{\text{Lt}} \frac{e^x + \log\left(\frac{1-x}{e}\right)}{\tan x - x}$

3. Evaluate the following limits :

(*i*) $\underset{x \to \infty}{\text{Lt}} \frac{a^{\frac{1}{x}} - b^{\frac{1}{x}}}{\log\frac{x}{x-1}}$

(*ii*) $\underset{x \to \infty}{\text{Lt}} \frac{2^{\frac{1}{x}} - 3^{\frac{1}{x}}}{\log\frac{x}{x-1}}$

4. Evaluate the following limits :

(*i*) $\underset{x \to 0}{\text{Lt}} \frac{1 - \cos x^2}{x^2 \sin x^2}$

(*ii*) $\underset{x \to 0}{\text{Lt}} \frac{\log(1+x^3)}{\sin^3 x}$

(*iii*) $\underset{x \to 0}{\text{Lt}} \frac{(e^x - 1)\tan^2 x}{x^3}$

5. Evaluate the following limits :

(*i*) $\underset{x \to \frac{\pi}{2}}{\text{Lt}} \frac{\cos x}{\frac{\pi}{2} - x}$

(*ii*) $\underset{x \to \pi}{\text{Lt}} \frac{1 + \cos x}{\tan^2 x}$

(*iii*) $\underset{x \to \pi+}{\text{Lt}} \frac{\sin x}{\sqrt{x - \pi}}$

(*iv*) $\underset{x \to 0+}{\text{Lt}} \frac{3^x - 2^x}{\sqrt{x}}$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

6. (*i*) If $\underset{x \to 0}{\text{Lt}} \dfrac{\sin 2x + a \sin x}{x^3}$ be finite, find the value of a and the limit.

(*ii*) Find the values of a and b so that

$$\underset{x \to 0}{\text{Lt}} \frac{x(1 - a\cos x) + b\sin x}{x^3} \text{ exists and equals to } \frac{1}{3}.$$

(*iii*) Given that $\underset{x \to 0}{\text{Lt}} \dfrac{\sin x + ax + bx^3}{x^5}$ is finite, find a and b. Find value of limit also.

(*iv*) Determine the values of a, b, c so that $\underset{x \to 0}{\text{Lt}} \dfrac{x(a + b\cos x) - c\sin x}{x^5}$ exists and equals to 1.

(*v*) Determine the values of a, b, c so that $\underset{x \to 0}{\text{Lt}} \dfrac{ae^x - b\cos x + ce^{-x}}{x\sin x}$ exists and equals 2.

(*vi*) If $\underset{y \to 0}{\text{Lt}} \dfrac{ae^y - b\cos y + ce^{-y}}{y\tan y} = 3$, find a, b, c.

7. Find a, b and the value of the limit, by using L′ Hospital's Rule if $\underset{x \to 0}{\text{Lt}} \dfrac{a\cos x + bx\sin x - 5}{x^4}$ exists and is finite.

8. Show that the following limits exist, but they cannot be evaluated by L' Hospital's Rule. What are the limits ?

(*i*) $\underset{x \to 1}{\text{Lt}} \dfrac{x^2 \sin \frac{1}{x}}{\sin x}$ (*ii*) $\underset{x \to 0}{\text{Lt}} \dfrac{\sin x^2 \cos \frac{1}{x}}{\cos x}$

(*iii*) $\underset{x \to \infty}{\text{Lt}} \dfrac{x - \sin x}{x}$

9. Evaluate the following limits :

(*i*) $\underset{x \to \infty}{\text{Lt}}\ x \sin \dfrac{1}{x}$ (*ii*) $\underset{x \to \infty}{\text{Lt}} \dfrac{\sin\left(\frac{2}{x}\right)}{\frac{1}{x}}$

(*iii*) $\underset{x \to \infty}{\text{Lt}} \dfrac{\sin \frac{1}{x}}{\tan^{-1}\left(\frac{1}{x}\right)}$ (*iv*) $\underset{x \to \infty}{\text{Lt}} \dfrac{\tan^{-1}\left(\frac{2}{x}\right)}{\frac{1}{x}}$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

10. What is the fallacy in the following ?

(*i*) $\underset{x \to 0}{\text{Lt}} \dfrac{6-x}{7+x} = \underset{x \to 0}{\text{Lt}} \dfrac{\frac{d}{dx}(6-x)}{\frac{d}{dx}(7+x)} = \underset{x \to 0}{\text{Lt}} \dfrac{-1}{1} = -1$

(*ii*) $\underset{x \to 1}{\text{Lt}} \dfrac{x^3+3x-4}{2x^2+x-3} = \underset{x \to 1}{\text{Lt}} \dfrac{3x^2+3}{4x+1} = \underset{x \to 1}{\text{Lt}} \dfrac{6x}{4} = \dfrac{3}{2}$.

ANSWERS

1. (*i*) $\dfrac{a}{b}$ (*ii*) 1 (*iii*) $\log_b a$ (*iv*) n (*v*) 1 (*vi*) -2

(*vii*) $\dfrac{1}{2}$ (*viii*) $\dfrac{1}{6}$ (*ix*) 2 (*x*) -1 (*xi*) $-\dfrac{2}{\pi}$ (*xii*) $-\dfrac{2}{3}$

(*xiii*) $\dfrac{1}{3}$ (*xiv*) 1 (*xv*) 2 (*xvi*) $\dfrac{1}{6}$ (*xvii*) 2 (*xviii*) 0

(*xix*) $\log(e\,a^{-1})\log(e\,a)^{-1}$ (*xx*) -2

2. (*i*) $\dfrac{1-\log b}{1+\log b}$ (*ii*) 2 (*iii*) $-\dfrac{1}{2}$

3. (*i*) $\log\left(\dfrac{a}{b}\right)$ (*ii*) $\log\left(\dfrac{2}{3}\right)$

4. (*i*) $\dfrac{1}{2}$ (*ii*) 1 (*iii*) 1

5. (*i*) 1 (*ii*) $\dfrac{1}{2}$ (*iii*) 0 (*iv*) 0

6. (*i*) $a=-2$, limit $=-1$ (*ii*) $a=\dfrac{1}{2}$, $b=-\dfrac{1}{2}$

(*iii*) $a=-1$, $b=\dfrac{1}{6}$; limit $=\dfrac{1}{120}$ (*iv*) $a=120$, $b=60$, $c=180$

(*v*) $a=1$, $b=2$, $c=1$ (*vi*) $a=\dfrac{3}{2}, b=3, c=\dfrac{3}{2}$

7. $a=5$, $b=\dfrac{5}{2}$ and limit $=-\dfrac{5}{24}$

8. (*i*) 0 (*ii*) 0 (*iii*) 1

9. (*i*) 1 (*ii*) 2 (*iii*) 1 (*iv*) 2

10. (*i*) Condition (*i*) of L' Hospital's Rule is not satisfied

(*ii*) Function is not of $\dfrac{0}{0}$ form

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Art-4. Indeterminate Form $\frac{\infty}{\infty}$

If f and g are two differentiable functions in the deleted *nbd.* N of a and $g'(x) \neq 0$ for all x in N, and

(*i*) $\underset{x \to a}{\text{Lt}} f(x) = \underset{x \to a}{\text{Lt}} g(x) = \infty$

(*ii*) $\underset{x \to a}{\text{Lt}} \frac{f'(x)}{g'(x)}$ exists, then $\underset{x \to a}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{g'(x)}$

Note 1. If $\underset{x \to a}{\text{Lt}} \frac{f'(x)}{g'(x)}$ again assumes the form $\frac{\infty}{\infty}$, then L' Hospital's Rule is repeated till the limit is found.

2. Above result holds when $x \to \infty$ or $x \to -\infty$.

ILLUSTRATIVE EXAMPLES

Example 1. Evaluate $\underset{x \to 0}{\text{Lt}} \frac{\log \sin x}{\cot x}$.

Sol. $\underset{x \to 0}{\text{Lt}} \frac{\log \sin x}{\cot x}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{\frac{1}{\sin x}.\cos x}{-\operatorname{cosec}^2 x} = -\underset{x \to 0}{\text{Lt}} \sin x \cos x = -\underset{x \to 0}{\text{Lt}} \sin x . \underset{x \to 0}{\text{Lt}} \cos x$$

$$= -0 \times 1 = 0.$$

Example 2. Evaluate $\underset{x \to 0}{\text{Lt}} \frac{\log x^2}{\cot x^2}$.

Sol. $\underset{x \to 0}{\text{Lt}} \frac{\log x^2}{\cot x^2}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{\frac{1}{x^2}.2x}{-2x \operatorname{cosec} x^2} = -\underset{x \to 0}{\text{Lt}} \frac{\sin x^2}{x^2} = -\underset{x \to 0}{\text{Lt}} \frac{2x \cos x^2}{2x}$$

$$= -\underset{x \to 0}{\text{Lt}} \cos x^2 = -1.$$

Example 3. Evaluate $\underset{x \to 0+}{\text{Lt}} \log_{\sin 2x} \sin x$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Sol. $\underset{x \to 0+}{\text{Lt}} \log_{\sin 2x} \sin x = \underset{x \to 0+}{\text{Lt}} \dfrac{\log \sin x}{\log \sin 2x}$ $\left(\dfrac{\infty}{\infty} \text{ form}\right)$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\dfrac{1}{\sin x}.\cos x}{\dfrac{1}{\sin 2x}.2\cos 2x} = \underset{x \to 0+}{\text{Lt}} \frac{\cos x}{\sin x} \times \frac{\sin 2x}{2\cos 2x}$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\cos x}{\sin x} \times \frac{2\sin x \cos x}{2\cos 2x}$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\cos^2 x}{\cos 2x} = \frac{1}{1}$$

$$= 1$$

Example 4. Evaluate $\underset{x \to \infty}{Lt} \dfrac{x^5}{e^x}$.

Sol. $\underset{x \to \infty}{Lt} \dfrac{x^5}{e^x}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to \infty}{Lt} \dfrac{5x^4}{e^x}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to \infty}{Lt} \dfrac{20x^3}{e^x}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to \infty}{Lt} \dfrac{60x^2}{e^x}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to \infty}{Lt} \dfrac{120x}{e^x}$ $\left(\dfrac{0}{0} \text{ form}\right)$

$= \underset{x \to \infty}{Lt} \dfrac{120}{e^x} = 120 \underset{x \to \infty}{Lt} \dfrac{1}{e^x}$

$= 120 \times 0$ $\left[\because \underset{x \to \infty}{Lt} \dfrac{1}{e^x} = \underset{x \to \infty}{Lt} e^{-x} = 0\right]$

$= 0$

Example 5. Evaluate $\underset{x \to \infty}{\text{Lt}} \dfrac{x^n}{e^x}$, $n \in \mathbf{N}$.

Sol. $\underset{x \to \infty}{\text{Lt}} \dfrac{x^n}{e^x}$ $\left(\dfrac{\infty}{\infty} \text{ form}\right)$

$= \underset{x \to \infty}{\text{Lt}} \dfrac{nx^{n-1}}{e^x}$ $\left(\dfrac{\infty}{\infty} \text{ form}\right)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$= \underset{x \to \infty}{\text{Lt}} \frac{n(n-1)x^{n-2}}{e^x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$= \underset{x \to \infty}{\text{Lt}} \frac{n(n-1)(n-2)\ldots 2.1}{e^x}$$

$$= \underset{x \to \infty}{\text{Lt}} \frac{\lfloor n}{e^x} = \lfloor n \cdot \underset{x \to \infty}{\text{Lt}} \frac{1}{e^x} = 0. \qquad \left[\because \underset{x \to \infty}{\text{Lt}} \frac{1}{e^x} = \underset{x \to \infty}{\text{Lt}} e^{-x} = 0\right]$$

EXERCISE 4 (*b*)

1. Evaluate the following limits

(*i*) $\underset{x \to \frac{\pi}{2}+}{\text{Lt}} \dfrac{\tan x}{\tan 3x}$ (*ii*) $\underset{x \to 1}{\text{Lt}} \dfrac{\log(1-x^2)}{\cot \pi x}$

(*iii*) $\underset{x \to 0}{\text{Lt}} \left(\dfrac{\log x^2}{\cot^2 x}\right)$ (*iv*) $\underset{x \to 0}{\text{Lt}} \log_{\tan^2 x} \tan^2 2x$.

2. Evaluate the following limits :

(*i*) $\underset{x \to \frac{\pi}{2}+}{\text{Lt}} \dfrac{\log\left(x - \frac{\pi}{2}\right)}{\tan x}$ (*ii*) $\underset{x \to 1-}{\text{Lt}} \log(1-x)\cot\left(\dfrac{\pi x}{2}\right)$

(*iii*) $\underset{x \to a+}{\text{Lt}} \dfrac{\log(x-a)}{\log(e^x - e^a)}$ (*iv*) $\underset{x \to 0+}{\text{Lt}} \log_{\tan x} \tan 2x$

3. Evaluate the following limits :

(*i*) $\underset{x \to \infty}{\text{Lt}} \dfrac{\log x}{x}$ (*ii*) $\underset{x \to \infty}{\text{Lt}} \dfrac{\log x}{x^n}$, $n \in \mathbf{N}$.

4. Point out the fallacy in the following :

$$\underset{x \to \infty}{\text{Lt}} \frac{x^2 - \cos x^2}{x^2} = \underset{x \to \infty}{\text{Lt}} \frac{2x + 2x\sin x^2}{2x} = \underset{x \to \infty}{\text{Lt}} (1 + \sin x^2)$$

and so the limit does not exist. Also find the limit.

ANSWERS

1. (*i*) 3 (*ii*) 0 (*iii*) 0 (*iv*) 1

2. (*i*) 0 (*ii*) 0 (*iii*) 1 (*iv*) 1

3. (*i*) 0 (*ii*) 0

4. $\underset{x \to \infty}{\text{Lt}} \dfrac{f'(x)}{g'(x)}$ does not exist where $f(x) = x^2 - \cos x^2$, $g(x) = x^2$, Limit = 1

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Art-5. Indeterminate Form $\infty - \infty$

If $\underset{x \to a}{\text{Lt}} f(x) = \underset{x \to a}{\text{Lt}} g(x) = \infty$, then to determine $\underset{x \to a}{\text{Lt}} [f(x) - g(x)]$, we write

$$f(x) - g(x) = \frac{\dfrac{1}{g(x)} - \dfrac{1}{f(x)}}{\dfrac{1}{f(x)g(x)}}$$ which is of the form $\dfrac{0}{0}$ as $x \to a$ and can be evaluated by using L' Hospital's Rule.

ILLUSTRATIVE EXAMPLES

Example 1. Evaluate $\underset{x \to 0}{\text{Lt}} \left(\dfrac{1}{e^x - 1} - \dfrac{1}{x} \right)$.

Sol. $\underset{x \to 0}{\text{Lt}} \left(\dfrac{1}{e^x - 1} - \dfrac{1}{x} \right)$ $\quad (\infty - \infty \text{ form})$

$$= \underset{x \to 0}{\text{Lt}} \left[\frac{x - e^x + 1}{x(e^x - 1)} \right] \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \left[\frac{1 - e^x + 0}{x.e^x + (e^x - 1).1} \right] = \underset{x \to 0}{\text{Lt}} \frac{1 - e^x}{xe^x + e^x - 1} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-e^x}{xe^x + e^x + e^x} = -\frac{1}{0 + 1 + 1} = -\frac{1}{2}.$$

Example 2. Evaluate $\underset{x \to 0}{\text{Lt}} \left(\cot^2 x - \dfrac{1}{x^2} \right)$.

Sol. $\underset{x \to 0}{\text{Lt}} \left(\cot^2 x - \dfrac{1}{x^2} \right)$ $\quad (\infty - \infty \text{ form})$

$$= \underset{x \to 0}{\text{Lt}} \left(\frac{1}{\tan^2 x} - \frac{1}{x^2} \right) = \underset{x \to 0}{\text{Lt}} \frac{x^2 - \tan^2 x}{x^2 \tan^2 x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x^2 - \tan^2 x}{x^4} \cdot \left(\frac{x}{\tan x} \right)^2$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x^2 - \tan^2 x}{x^4} \cdot (1)^2. \quad \left[\because \underset{x \to 0}{\text{Lt}} \frac{x}{\tan x} = 1 \right]$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$= \underset{x \to 0}{\text{Lt}} \frac{x^2 - \tan^2 x}{x^4} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2x - 2\tan x \sec^2 x}{4x^3} = \underset{x \to 0}{\text{Lt}} \frac{2x - 2\tan x(1 + \tan^2 x)}{4x^3}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x - \tan x - \tan^3 x}{2x^3} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{1 - \sec^2 x - 3\tan^2 x \sec^2 x}{6x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{1 - (1 + \tan^2 x) - 3\tan^2 x(1 + \tan^2 x)}{6x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{1 - 1 - \tan^2 x - 3\tan^2 x - 3\tan^4 x}{6x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-4\tan^2 x - 3\tan^4 x}{6x^2} = -\underset{x \to 0}{\text{Lt}} \frac{4 + 3\tan^2 x}{6} \cdot \left(\frac{\tan x}{x}\right)^2$$

$$= -\frac{4+0}{6} \cdot (1)^2 = -\frac{2}{3}.$$

EXERCISE 4 (c)

1. Evaluate the following limits :

(*i*) $\underset{x \to 1}{\text{Lt}} \left(\frac{1}{x^2 - 1} - \frac{2}{x^4 - 1}\right)$ (*ii*) $\underset{x \to 0}{\text{Lt}} \left[\frac{1}{2x} - \frac{1}{x(e^{\pi x} + 1)}\right]$

(*iii*) $\underset{x \to 2}{\text{Lt}} \left[\frac{1}{\log(x-1)} - \frac{1}{x-2}\right]$ (*iv*) $\underset{x \to 1}{\text{Lt}} \left(\frac{x}{x-1} - \frac{1}{\log x}\right)$

(*v*) $\underset{x \to 0}{\text{Lt}} \left[\frac{1}{x} - \frac{1}{x^2}\log(1+x)\right]$

2. Evaluate the following limits :

(*i*) $\underset{x \to 0}{\text{Lt}} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$ (*ii*) $\underset{x \to 0}{\text{Lt}} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

(*iii*) $\underset{x \to 0}{\text{Lt}} \left(\frac{1}{x} - \cot x\right)$ (*iv*) $\underset{x \to 0}{\text{Lt}} \left(\frac{1}{x^2} - \frac{\cot x}{x}\right)$

(*v*) $\underset{x \to \frac{\pi}{2}}{\text{Lt}} (\sec x - \tan x)$ (*vi*) $\underset{x \to 0}{\text{Lt}} (\text{cosec}\, x - \cot x)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

3. Evaluate $\underset{x \to \frac{\pi}{2}}{\text{Lt}} \left(x \tan x - \frac{\pi}{2} \sec x \right)$

ANSWERS

1. (*i*) $\frac{1}{2}$ (*ii*) $\frac{\pi}{4}$ (*iii*) $\frac{1}{2}$ (*iv*) $\frac{1}{2}$ (*v*) $\frac{1}{2}$

2. (*i*) $-\frac{1}{3}$ (*ii*) 0 (*iii*) 0 (*iv*) $\frac{1}{3}$ (*v*) 0 (*vi*) 0

3. -1

Art-6. Indeterminate Form 0 . ∞

If $\underset{x \to a}{\text{Lt}} f(x) = 0$ and $\underset{x \to a}{\text{Lt}} g(x) = \infty$, then to find $\underset{x \to a}{\text{Lt}} f(x)\, g(x)$ we write $f(x)\, g(x) = \dfrac{f(x)}{\frac{1}{g(x)}}$ or $\dfrac{g(x)}{\frac{1}{f(x)}}$ which are of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ as $x \to a$ and can be evaluated by using L' Hospital's Rule.

The form 0 . (– ∞) is also evaluated in the same manner.

ILLUSTRATIVE EXAMPLES

Example 1. Evaluate $\underset{x \to 0}{\text{Lt}} \sin x \log x^2$

Sol. $\underset{x \to 0}{\text{Lt}} (\sin x \log x^2)$ (0 . ∞ form)

$$= \underset{x \to 0}{\text{Lt}} \frac{\log x^2}{\frac{1}{\sin x}} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\log x^2}{\operatorname{cosec} x} = \underset{x \to 0}{\text{Lt}} \frac{\frac{1}{x^2}.2x}{-\operatorname{cosec} x \cot x} = -2 \underset{x \to 0}{\text{Lt}} \left(\frac{\sin^2 x}{x^2} . \frac{x}{\cos x} \right)$$

$$= -2 \underset{x \to 0}{\text{Lt}} \left(\frac{\sin x}{x} \right)^2 . \underset{x \to 0}{\text{Lt}} \frac{1}{\cos x} . \underset{x \to 0}{\text{Lt}}\, x = -2\,(1) \times \frac{1}{1} \times 0$$

$$= 0.$$

Example 2. Evaluate $\underset{x \to 1}{\text{Lt}} (1 - x) \tan \frac{\pi x}{2}$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Sol. $\underset{x \to 1}{\text{Lt}} (1-x)\tan\frac{\pi x}{2}$ (0 . ∞ form)

$$= \underset{x \to 1}{\text{Lt}} \frac{1-x}{\cot\frac{\pi x}{2}} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 1}{\text{Lt}} \frac{-1}{\left(-\text{cosec}^2\frac{\pi x}{2}\right)\times\frac{\pi}{2}} = \frac{2}{\pi}.\sin^2\frac{\pi}{2} = \frac{2}{\pi}\times(1) = \frac{2}{\pi}.$$

Example 3. Evaluate $\underset{x \to 0+}{\text{Lt}} x^m . \log x$, where $m > 0$.

Sol. $\underset{x \to 0+}{\text{Lt}} x^m . \log x$ (0 . ∞ form)

$$= \underset{x \to 0+}{\text{Lt}} \frac{\log x}{x^{-m}} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\frac{1}{x}}{-m x^{-m-1}} = -\frac{1}{m} \underset{x \to 0+}{\text{Lt}} x^m = 0$$

EXERCISE 4 (*d*)

1. Evaluate the following limits :

(*i*) $\underset{x \to \frac{\pi}{2}}{\text{Lt}} (1-\sin x)\tan x$ (*ii*) $\underset{x \to 1}{\text{Lt}} \sec\frac{\pi x}{2}\log\frac{1}{x}$

(*iii*) $\underset{x \to 0}{\text{Lt}} x\tan\left(\frac{\pi}{2}-x\right)$ (*iv*) $\underset{x \to c}{\text{Lt}} (c-x)\tan\left(\frac{\pi x}{2c}\right)$

(*v*) $\underset{x \to \infty}{\text{Lt}} x\tan\frac{1}{x}$ (*vi*) $\underset{x \to \infty}{\text{Lt}} 2^x \sin\left(\frac{a}{2^x}\right)$

2. Evaluate the following limits :

(*i*) $\underset{x \to 0+}{\text{Lt}} x\log x$ (*ii*) $\underset{x \to 0+}{\text{Lt}} x^m (\log x)^n$, $m, n \in \mathbb{N}$

(*iii*) $\underset{x \to 0+}{\text{Lt}} x\log\tan x$ (*iv*) $\underset{x \to 0+}{\text{Lt}} \sin x . \log x$

ANSWERS

1. (*i*) 0 (*ii*) $\frac{2}{\pi}$ (*iii*) 1 (*iv*) $\frac{2c}{\pi}$ (*v*) 1 (*vi*) a

2. (*i*) 0 (*ii*) 0 (*iii*) 0 (*iv*) 0

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Art-7. Indeterminate Forms 0^0, 1^∞, ∞^0

Here we are to evaluate $\underset{x \to a}{\text{Lt}} [f(x)]^{g(x)}$, when

(*i*) $\underset{x \to a}{\text{Lt}} f(x) = 0 = \underset{x \to a}{\text{Lt}} g(x)$ (*ii*) $\underset{x \to a}{\text{Lt}} f(x) = 1$, $\underset{x \to a}{\text{Lt}} g(x) = \infty$

(*iii*) $\underset{x \to a}{\text{Lt}} f(x) = \infty$, $\underset{x \to a}{\text{Lt}} g(x) = 0$

We proceed like this :

Let $y = [f(x)]^{g(x)}$

Taking logs on both sides,

$$\log y = \log [f(x)]^{g(x)} \Rightarrow \log y = g(x) \,.\, \log [f(x)]$$

Now $g(x) \,.\, \log f(x)$ becomes of the form $0 \,.\, \infty$ when $x \to a$ and can be evaluated.

Let $\underset{x \to a}{\text{Lt}} g(x) \,.\, \log f(x) = l$

$$\therefore \underset{x \to a}{\text{Lt}} \log y = l \Rightarrow \log \underset{x \to a}{\text{Lt}} y = l$$

$$\Rightarrow \underset{x \to a}{\text{Lt}} y = e^l \Rightarrow \underset{x \to a}{\text{Lt}} [f(x)]^{g(x)}] = e^l.$$

ILLUSTRATIVE EXAMPLES

Example 1. Evaluate $\underset{x \to 0+}{\text{Lt}} x^x$.

Sol. Let $y = x^x$

$$\therefore \log y = \log x^x \Rightarrow \log y = x \,.\, \log x$$

$$\therefore \underset{x \to 0+}{\text{Lt}} \log y = \underset{x \to 0+}{\text{Lt}} x \,.\, \log x \qquad (0 \,.\, \infty \text{ form})$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\log x}{\frac{1}{x}} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \underset{x \to 0+}{\text{Lt}} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \underset{x \to 0+}{\text{Lt}} (-x) = 0$$

$$\therefore \log \underset{x \to 0+}{\text{Lt}} y = 0 \Rightarrow \underset{x \to 0+}{\text{Lt}} y = e^0$$

$$\Rightarrow \underset{x \to 0+}{\text{Lt}} x^x = 1.$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 2. Evaluate $\underset{x \to 0}{\text{Lt}} (\sin x)^{\tan x}$.

Sol. Let $y = (\sin x)^{\tan x}$

$\therefore \log y = \log (\sin x)^{\tan x} \Rightarrow \log y = \tan x \,.\, \log (\sin x)$

$$\therefore \underset{x \to 0}{\text{Lt}} \log y = \underset{x \to 0}{\text{Lt}} [\tan x \,.\, \log (\sin x)] = \underset{x \to 0}{\text{Lt}} \frac{\log (\sin x)}{\cot x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\frac{1}{\sin x} . \cos x}{-\text{cosec}^2 x}$$

$$= -\underset{x \to 0}{\text{Lt}} (\sin x \cos x) = -(0 \times 1)$$

$\because \underset{x \to 0}{\text{Lt}} \log y = 0 \quad \Rightarrow \log \underset{x \to 0}{\text{Lt}} y = 0$

$\Rightarrow \underset{x \to 0}{\text{Lt}} y = e^0 \quad \Rightarrow \underset{x \to 0}{\text{Lt}} (\sin x)^{\tan x} = 1.$

Example 3. Evaluate $\underset{x \to 0}{\text{Lt}} \left(\frac{1}{x^2}\right)^{\tan x}$.

Sol. Let $y = \left(\frac{1}{x^2}\right)^{\tan x}$

$\therefore \log y = \log \left(\frac{1}{x^2}\right)^{\tan x} \Rightarrow \log y = \tan x \,[\log 1 - \log x^2]$

$\Rightarrow \log y = \tan x \,[0 - 2 \log x] \Rightarrow \log y = -2 \tan x \log x$

$$\therefore \underset{x \to 0}{\text{Lt}} \log y = -2 \underset{x \to 0}{\text{Lt}} \tan x \,.\, \log x \quad (0 \,.\, \infty \text{ form})$$

$$= -2 \underset{x \to 0}{\text{Lt}} \frac{\log x}{\cot x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= -2 \underset{x \to 0}{\text{Lt}} \frac{\frac{1}{x}}{-\text{cosec}^2 x} = 2 \underset{x \to 0}{\text{Lt}} \frac{\sin^2 x}{x} = 2 \underset{x \to 0}{\text{Lt}} \left(\frac{\sin x}{x}\right)^2 . x$$

$$= 2\,(1)^2 \,.\, 0$$

$\therefore \underset{x \to 0}{\text{Lt}} \log y = 0 \quad \Rightarrow \log \underset{x \to 0}{\text{Lt}} y = 0$

$\Rightarrow \underset{x \to 0}{\text{Lt}} y = e^0 \quad \Rightarrow \underset{x \to 0}{\text{Lt}} \left(\frac{1}{x^2}\right)^{\tan x} = 1.$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 4. Evaluate $\underset{x \to 0}{\text{Lt}} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$.

Sol. Let $y = \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

$$\therefore \quad \log y = \log \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} \quad \Rightarrow \quad \log y = \frac{1}{x^2} \log \left(\frac{\sin x}{x}\right)$$

$$\therefore \quad \underset{x \to 0}{\text{Lt}} \log y = \underset{x \to 0}{\text{Lt}} \frac{1}{x^2} . \log \left(\frac{\sin x}{x}\right) \qquad (0 \,.\, \infty \text{ form})$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\log \left(\frac{\sin x}{x}\right)}{x^2} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\frac{x}{\sin x} \times \frac{x \cos x - \sin x}{x^2}}{2x} = \underset{x \to 0}{\text{Lt}} \frac{x \cos x - \sin x}{2x^2 \sin x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x \cos x - \sin x}{2x^3 \left(\frac{\sin x}{x}\right)} = \frac{1}{2} \underset{x \to 0}{\text{Lt}} \frac{x \cos x - \sin x}{x^3} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \frac{1}{2} \underset{x \to 0}{\text{Lt}} \frac{-x \sin x + \cos x - \cos x}{3x^2} = -\frac{1}{6} \underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = -\frac{1}{6} (1)$$

$$\therefore \quad \underset{x \to 0}{\text{Lt}} \log y = -\frac{1}{6} \qquad \Rightarrow \quad \log \underset{x \to 0}{\text{Lt}} y = -\frac{1}{6}$$

$$\Rightarrow \quad \underset{x \to 0}{\text{Lt}} y = e^{-\frac{1}{6}} \qquad \Rightarrow \quad \underset{x \to 0}{\text{Lt}} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = e^{-\frac{1}{6}}.$$

Example 5. Evaluate $\underset{x \to 0}{\text{Lt}} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$. (J.U. 2016)

Sol. Let $y = \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

$$\therefore \quad \log y = \log \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} \quad \Rightarrow \quad \log y = \frac{1}{x^2} \log \left(\frac{\tan x}{x}\right)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\therefore \quad \underset{x\to 0}{\text{Lt}} \log y = \underset{x\to 0}{\text{Lt}} \frac{\log\left(\frac{\tan x}{x}\right)}{x^2} = \underset{x\to 0}{\text{Lt}} \frac{\frac{x}{\tan x}\cdot\frac{x\sec^2 x-\tan x}{x^2}}{2x}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{x\sec^2 x-\tan x}{2x^3} \qquad \left[\because \underset{x\to 0}{\text{Lt}} \frac{x}{\tan x}=1\right]$$

$$= \underset{x\to 0}{\text{Lt}} \frac{x\,.\,2\sec x\,.\,\sec x\tan x+\sec^2 x-\sec^2 x}{6x^2}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{2x\sec^2 x\tan x}{6x^2} = \frac{1}{3}\underset{x\to 0}{\text{Lt}}\left(\frac{1}{\cos^2 x}\cdot\frac{\tan x}{x}\right)$$

$$= \frac{1}{3}\left(\frac{1}{1}\times 1\right) = \frac{1}{3}$$

$$\therefore \quad \underset{x\to 0}{\text{Lt}} \log y = \frac{1}{3} \quad \Rightarrow \quad \log \underset{x\to 0}{\text{Lt}}\, y = \frac{1}{3} \quad \Rightarrow \quad \underset{x\to 0}{\text{Lt}}\, y = e^{\frac{1}{3}}$$

$$\therefore \quad \underset{x\to 0}{\text{Lt}} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = e^{\frac{1}{3}}$$

Example 6. Evaluate the following limit :

$$\underset{x\to 0}{\text{Lt}} \left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}$$

Sol. Let $\quad y = \left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}$

$$\therefore \quad \log y = \log\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}} = \frac{1}{x}\log\left(\frac{a^x+b^x}{2}\right) = \frac{1}{x}\left[\log(a^x+b^x)-\log 2\right]$$

$$\therefore \quad \underset{x\to 0}{\text{Lt}} \log y = \underset{x\to 0}{\text{Lt}} \frac{\log(a^x+b^x)-\log 2}{x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\frac{a^x\log a+b^x\log b}{a^x+b^x}-0}{1} = \frac{\log a+\log b}{1+1} = \frac{1}{2}\log(a\,b)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\therefore \quad \underset{x \to 0}{\text{Lt}} \log \left(\frac{a^x + b^x}{2} \right)^{\frac{1}{x}} = \log (ab)^{\frac{1}{2}}$$

$$\Rightarrow \quad \log \underset{x \to 0}{\text{Lt}} \left(\frac{a^x + b^x}{2} \right)^{\frac{1}{x}} = \log (ab)^{\frac{1}{2}} \quad \Rightarrow \quad \underset{x \to 0}{\text{Lt}} \left(\frac{a^x + b^x}{2} \right)^{\frac{1}{x}} = \sqrt{ab}$$

EXERCISE 4 (*e*)

1. Evaluate $\underset{x \to 0}{\text{Lt}} \left[\tan \left(\frac{\pi}{4} + x \right) \right]^{\frac{1}{x}}$.

2. Evaluate the following limits :

(*i*) $\underset{x \to 0+}{\text{Lt}} \; x^{\sin x}$ (*ii*) $\underset{x \to 0+}{\text{Lt}} \; x^{\frac{1}{\log x}}$

(*iii*) $\underset{x \to a+}{\text{Lt}} \; (x - a)^{x - a}$ (*iv*) $\underset{x \to 1-}{\text{Lt}} \; (1 - x^2)^{\frac{1}{\log (1-x)}}$

(*v*) $\underset{x \to 0+}{\text{Lt}} \; (\tan x)^{\sin 2x}$ (*vi*) $\underset{x \to \frac{\pi}{2}-}{\text{Lt}} \; (\cos x)^{\cos x}$

3. Evaluate the following limits :

(*i*) $\underset{x \to \infty}{\text{Lt}} \; (1 + x)^{\frac{1}{x}}$ (*ii*) $\underset{x \to 0+}{\text{Lt}} \; (\cot x)^{x}$

(*iii*) $\underset{x \to 0+}{\text{Lt}} \; (\cot x)^{\sin x}$ (*iv*) $\underset{x \to \frac{\pi}{2}-}{\text{Lt}} \; (\tan x)^{\sin 2x}$

(*v*) $\underset{x \to 0+}{\text{Lt}} \left(\frac{1}{x} \right)^{\tan x}$ (*vi*) $\underset{x \to 0+}{\text{Lt}} \; (\cot x)^{\frac{1}{\log x}}$

(*vii*) $\underset{x \to \frac{\pi}{2}-}{\text{Lt}} \; (\sec x)^{\cot x}$ (*viii*) $\underset{x \to 0+}{\text{Lt}} \; (\cot x)^{\sin 2x}$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

4. Evaluate the following limits :

(*i*) $\underset{x \to 0}{\text{Lt}} (1+x)^{\frac{1}{x}}$ (*ii*) $\underset{x \to \frac{\pi}{4}}{\text{Lt}} (\tan x)^{\tan 2x}$

(*iii*) $\underset{x \to 0}{\text{Lt}} (1+\sin x)^{\cot x}$ (*iv*) $\underset{x \to 1}{\text{Lt}} x^{\frac{1}{x-1}}$

(*v*) $\underset{x \to \infty}{\text{Lt}} \left(1+\frac{a}{x}\right)^{x}$ (*vi*) $\underset{x \to 0}{\text{Lt}} (e^x + 4x)^{\frac{1}{x}}$

(*vii*) $\underset{x \to a}{\text{Lt}} \left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$ (*viii*) $\underset{x \to 0}{\text{Lt}} \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

5. Evaluate $\underset{x \to 0}{\text{Lt}} \left(\frac{a_1^{\,x} + a_2^{\,x} + \ldots + a_n^{\,x}}{n}\right)^{\frac{1}{x}}$.

ANSWERS

1. e^2

2. (*i*) 1 (*ii*) e (*iii*) 1 (*iv*) e (*v*) 1 (*vi*) 1

3. (*i*) 1 (*ii*) 1 (*iii*) 1 (*iv*) 1 (*v*) 1 (*vi*) $\frac{1}{e}$

(*vii*) 1 (*viii*) 1

4. (*i*) e (*ii*) $\frac{1}{e}$ (*iii*) e (*iv*) e (*v*) e^a (*vi*) e^5

(*vii*) $e^{2/\pi}$ (*viii*) 1

5. $(a_1 . a_2 . \ldots . a_n)^{\frac{1}{n}}$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

5

PARTIAL DIFFERENTIATION

Art-1. Introduction

Let $z = f(x, y)$ be a function of two independent variables x and y. Then the partial derivative of z with respect to x is the ordinary derivative of z y when y is regarded as a constant. Similarly the partial derivative of z *w.r.t.* y is the ordinary derivative of z when x is regarded as a constant.

For example if

$$z = 3x^3y^2 + 5x^2y^3,$$

then partial derivation of z *w. r. t.* x

$$= 3y^2(3x^{3-1}) + 5y^3(2x^{2-1}) = 9y^2x^2 + 10y^3x$$

and partial derivative of z *w. r. t.* y

$$= 3x^3(2y^{2-1}) + 5x^2(3y^{3-1}) = 6x^3y + 15x^2y^2.$$

On the same lines, if $z = f(x, x_2, \ldots, x_n)$ is a function of n independent variables $x_1, x_2, \ldots\ldots, x_n$, then the partial derivative of z with respect to any one of the variables is the ordinary derivative of z *w.r.t.* that variable when other variables are regarded as constant.

Art-2. First Order Partial Derivatives

Let $z = f(x, y)$ be a real valued function of two real independent variables with an open domain $D_f \subset \mathbf{R}^2$, then

$$\underset{\delta x \to 0}{\text{Lt}} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

if it exists, is called the *partial derivative of f w.r.t. x.* It is denoted by

$$\frac{\partial z}{\partial x} \text{ or } \frac{\partial f}{\partial x} \text{ or } f_x \text{ or } f_x(x, y) \text{ or } D_1 f.$$

$$\therefore \quad \frac{\partial f}{\partial x} = \underset{\delta x \to 0}{\text{Lt}} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

Also $\underset{\delta y \to 0}{\text{Lt}} \dfrac{f(x, y+\delta y) - f(x, y)}{\delta y}$, if it exists, is called the partial *derivative of* f *w.r.t.* y. It is denoted by $\dfrac{\partial z}{\partial y}$ or $\dfrac{\partial f}{\partial y}$ or f_y or $f_y(x, y)$ or $D_2 f$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Thus

$$\frac{\partial f}{\partial y} = \underset{\delta y \to 0}{\text{Lt}} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}$$

The partial derivative of f w.r.t. x at any point $(a, b) \in D_f$ is denoted by

$$\left(\frac{\partial z}{\partial x}\right)_{(a,b)} \text{ or } \left(\frac{\partial f}{\partial x}\right)_{(a,b)} \text{ or } f_x(a, b). \text{ Thus}$$

$$f_x(a, b) = \underset{h \to 0}{\text{Lt}} \frac{f(a+h, b) - f(a, b)}{h}$$

Similarly the partial derivative of f w.r.t. y at any point $(a, b) \in D_f$ is denoted by

$$\left(\frac{\partial z}{\partial y}\right)_{(a,b)} \text{ or } \left(\frac{\partial f}{\partial y}\right)_{(a,b)} \text{ or } f_y(a, b).$$

Thus $$f_y(a, b) = \underset{k \to 0}{\text{Lt}} \frac{f(a, b+k) - f(a, k)}{k}$$

Art-3. Partial Derivatives of Second Order

If $z = f(x, y)$ and the first order partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist, then they are themselves functions of x and y. Their partial derivatives w.r.t. x and y, if they exist, are called second order partial derivatives of $f(x, y)$. We have

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \quad \text{or } f_x^2 \quad \text{or } f_{xx} \quad \text{or } \frac{\partial^2 z}{\partial x^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \quad \text{or } f_{xy} \quad \text{or } \frac{\partial^2 z}{\partial y \partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \quad \text{or } f_{yx} \quad \text{or } \frac{\partial^2 z}{\partial x \partial y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} \quad \text{or } f_y^2 \quad \text{or } f_{yy} \quad \text{or } \frac{\partial^2 z}{\partial y^2}$$

Thus,

$$\frac{\partial^2 f}{\partial x^2} = \underset{\delta x \to 0}{\text{Lt}} \frac{f_x(x+\delta x, y) - f_x(x, y)}{\delta x}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \underset{\delta y \to 0}{\text{Lt}} \frac{f_x(x, y+\delta y) - f_x(x, y)}{\delta y}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\frac{\partial^2 f}{\partial x\,\partial y} = \underset{\delta x \to 0}{\text{Lt}} \frac{f_y(x+\delta x, y) - f_y(x, y)}{\delta x}$$

$$\frac{\partial^2 f}{\partial y^2} = \underset{\delta y \to 0}{\text{Lt}} \frac{f_y(x, y+\delta y) - f_y(x, y)}{\delta y}$$

ILLUSTRATIVE EXAMPLES

Example 1. Let $f(x, y) = \log(xy + 2y^2 - 2x)$. Find $f_x(2, 3)$ and $f_y(2, 3)$.

Sol. Here $f(x, y) = \log(xy + 2y^2 - 2x)$

$$\therefore \quad f_x = \frac{\partial}{\partial x}\left[\log(xy + 2y^2 - 2x)\right]$$

$$= \frac{1}{xy + 2y^2 - 2x} \times (y - 2) = \frac{y-2}{xy + 2y^2 - 2x}$$

$$\therefore \quad f_x(2, 3) = \frac{3-2}{(2)(3) + 2(3)^2 - 2(2)} = \frac{1}{6+18-4} = \frac{1}{20}$$

Again $\quad f_y = \frac{\partial}{\partial y}\left[\log(xy + 2y^2 - 2x)\right] = \frac{x + 4y}{xy + 2y^2 - 2x}$

$$\therefore \quad f_y(2, 3) = \frac{2+12}{6+18-4} = \frac{14}{20} = \frac{7}{10}.$$

Example 2. If $z = x^n \log\left(\frac{y}{x}\right)$, verify that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$

Sol. Here $z = x^n \log\left(\frac{y}{x}\right)$ ...(1)

$$\therefore \quad \frac{\partial z}{\partial x} = nx^{n-1}\log\left(\frac{y}{x}\right) + x^n \cdot \frac{1}{\frac{y}{x}} \cdot \left(-\frac{y}{x^2}\right) = nx^{n-1}\log\left(\frac{y}{x}\right) - x^{n-1}$$

and $\quad \frac{\partial z}{\partial y} = x^n \cdot \frac{1}{\frac{y}{x}} \cdot \frac{1}{x} = \frac{x^n}{y}$

$$\therefore \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left[nx^{n-1}\log\left(\frac{y}{x}\right) - x^{n-1}\right] + y\left(\frac{x^n}{y}\right)$$

$$= nx^n \log\left(\frac{y}{x}\right) - x^n + x^n = nx^n \log\left(\frac{y}{x}\right)$$

$$\therefore \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \qquad [\because \text{ of } (1)]$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College of Education Banatalab Jammu

Example 3. If $z = \log\,(x^2 + xy + y^2)$, then prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2.$

Sol. Here $z = \log\,(x^2 + xy + y^2)$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{1}{x^2+xy+y^2}\times(2x+y) = \frac{2x+y}{x^2+xy+y^2}$$

$$\therefore \quad x\frac{\partial z}{\partial x} = \frac{2x^2+xy}{x^2+xy+y^2} \qquad \ldots(1)$$

Again $$\frac{\partial z}{\partial y} = \frac{1}{x^2+xy+y^2}\times(x+2y) = \frac{x+2y}{x^2+xy+y^2}$$

$$\therefore \quad y\frac{\partial z}{\partial y} = \frac{xy+2y^2}{x^2+xy+y^2} \qquad \ldots(2)$$

Adding (1) and (2), we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{2x^2+xy}{x^2+xy+y^2} + \frac{xy+2y^2}{x^2+xy+y^2} = \frac{2x^2+xy+xy+2y^2}{x^2+xy+y^2}$$

$$= \frac{2x^2+2xy+2y^2}{x^2+xy+y^2} = \frac{2(x^2+xy+y^2)}{x^2+xy+y^2}$$

$$\therefore \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2.$$

Example 4. (*a*) If $u = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = -u$

(*b*) If $u = \sqrt{x^2+y^2+z^2}$, then prove that $\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 + \left(\dfrac{\partial u}{\partial z}\right)^2 = 1$

Sol. (*a*) Here $u\,(x^2 + y^2 + z^2)^{-\frac{1}{2}}$

$$\therefore \quad \frac{\partial u}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot 2x = -x(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\therefore \quad \frac{\partial u}{\partial x} = -\frac{x}{(x^2+y^2+z)^{3/2}}$$

$$\therefore \quad x\frac{\partial u}{\partial x} = -\frac{x^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} \qquad \ldots(1)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Similarly $y\dfrac{\partial u}{\partial y} = -\dfrac{y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$ ...(2)

and $z\dfrac{\partial u}{\partial z} = -\dfrac{z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$ ...(3)

Adding (1), (2) and (3), we get,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = -\frac{x^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}-\frac{y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}-\frac{z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$= -\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} = -\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$= -(x^2+y^2+z^2)^{-\frac{1}{2}} = -u$$

$$\therefore \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = -u.$$

(*b*) Here $u = (x^2+y^2+z^2)^{\frac{1}{2}}$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}, 2x$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow \quad \left(\frac{\partial u}{\partial x}\right)^2 = \frac{x^2}{x^2+y^2+z^2} \quad ...(1)$$

Similarly $\left(\dfrac{\partial u}{\partial y}\right)^2 = \dfrac{y^2}{x^2+y^2+z^2}$ ...(2)

and $\left(\dfrac{\partial u}{\partial z}\right)^2 = \dfrac{z^2}{x^2+y^2+z^2}$ ...(3)

Adding (1), (2) and (3), we get

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 = \frac{x^2}{x^2+y^2+z^2}+\frac{y^2}{x^2+y^2+z^2}+\frac{z^2}{x^2+y^2+z^2}$$

$$= \frac{x^2+y^2+z^2}{x^2+y^2+z^2}$$

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 = 1$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 5. If $\theta = t^n \cdot e^{\frac{-r^2}{4t}}$, find the value of n which will make

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t}.$$

Sol. Here $\theta = t^n \cdot e^{\frac{-r^2}{4t}}$

$$\therefore \quad \frac{\partial\theta}{\partial t}=t^n.\frac{\partial}{\partial t}\left(e^{-\frac{r^2}{4t}}\right)+e^{-\frac{r^2}{4t}}.\frac{\partial}{\partial t}(t^n) = t^n.e^{-\frac{r^2}{4t}}.\frac{r^2}{4t^2}+e^{-\frac{r^2}{4t}}.nt^{n-1}$$

$$\therefore \quad \frac{\partial\theta}{\partial t}=\left(\frac{r^2t^{n-2}}{4}+nt^{n-1}\right)e^{-\frac{r^2}{4t}} \qquad ...(1)$$

Again $\quad \dfrac{\partial\theta}{\partial r}=t^n.e^{-\frac{r^2}{4t}}.\dfrac{-2r}{4t}=-\dfrac{1}{2}rt^{n-1}e^{-\frac{r^2}{4t}}$

$$\therefore \quad r^2\frac{\partial\theta}{\partial r}=-\frac{1}{2}r^3t^{n-1}e^{-\frac{r^2}{4t}}$$

$$\therefore \quad \frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=-\frac{1}{2}t^{n-1}\frac{\partial}{\partial r}\left[r^3.e^{-\frac{r^2}{4t}}\right]$$

$$=-\frac{1}{2}t^{n-1}\left[r^3.e^{-\frac{r^2}{4t}}.\frac{-2r}{4t}+e^{-\frac{r^2}{4t}}.3r^2\right]$$

$$=-\frac{1}{2}t^{n-1}\left[-\frac{1}{2}\frac{r^4}{t}+3r^2\right]e^{-\frac{r^2}{4t}}$$

$$\therefore \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=-\frac{1}{2}t^{n-1}\left[-\frac{1}{2}\frac{r^2}{t}+3\right]e^{-\frac{r^2}{4t}} \qquad ...(2)$$

Now $\quad \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\theta}{\partial r}\right)=\dfrac{\partial\theta}{\partial t}$

$$\Rightarrow \quad -\frac{1}{2}t^{n-1}\left[-\frac{1}{2}\frac{r^2}{t}+3\right]e^{-\frac{r^2}{4t}}=\left(\frac{r^2}{4}t^{n-2}+nt^{n-1}\right)e^{-\frac{r^2}{4t}} \qquad [\because \text{ of (1) and (2)}]$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\Rightarrow \quad -\frac{1}{2}t^{n-1}\left[-\frac{r^2}{2t}+3\right]e^{-\frac{r^2}{4t}} = t^{n-2}\left(\frac{r^2}{4}+nt\right)e^{-\frac{r^2}{4t}}$$

$$\Rightarrow \quad -\frac{1}{2}\left[-\frac{r^2}{2t}+3\right]e^{-\frac{r^2}{4t}} = t^{n-2}\left(\frac{r^2}{4}+nt\right)e^{-\frac{r^2}{4t}}$$

$$\Rightarrow \quad -\frac{1}{2}\left[-\frac{r^2}{2t}+3\right] = \frac{r^2}{4}+nt \qquad \left(\text{Dividing both sides by } t^{n-2}e^{-\frac{r^2}{4t}}\right)$$

$$\Rightarrow \quad \frac{r^2}{4}-\frac{3t}{2} = \frac{r^2}{4}+nt$$

$$\Rightarrow \quad \frac{-3t}{2} = nt \qquad \Rightarrow n = -\frac{3}{2}$$

Example 6. Given $u = e^x \cos y + e^y \sin z$. Find all first and second order partial derivatives.

Sol. $u = e^x \cos y + e^y \sin z$

$$\therefore \quad \frac{\partial u}{\partial x} = e^x \cos y + 0 = e^x \cos y, \quad \frac{\partial u}{\partial y} = -e^x \sin y + e^y \sin z$$

$$\frac{\partial u}{\partial z} = 0 + e^y \cos z = e^y \cos z$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y, \quad \frac{\partial^2 u}{\partial y \partial x} = -e^x \sin y, \quad \frac{\partial^2 u}{\partial z \partial x} = 0$$

$$\frac{\partial^2 u}{\partial y^2} = -e^x \cos y + e^y \sin z, \quad \frac{\partial^2 u}{\partial x \partial y} = -e^x \sin y + 0 = -e^x \sin y$$

$$\frac{\partial^2 x}{\partial z \partial y} = 0 + e^y \cos z = e^y \cos z$$

$$\frac{\partial^2 u}{\partial z^2} = -e^y \sin z, \quad \frac{\partial^2 u}{\partial x \partial z} = 0, \quad \frac{\partial^2 u}{\partial y \partial z} = e^y \cos z.$$

Example 7. If $z = f(x + ct) + \phi(x - ct)$, then prove that $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$.

Sol. Here $z = f(x + ct) + \phi(x - ct)$

$$\frac{\partial z}{\partial x} = f'(x+ct) + \phi'(x-ct)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct) \qquad \ldots(1)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Again $\dfrac{\partial z}{\partial t} = f'(x+ct).(c)+\phi'(x-ct).(-c)$

$$\frac{\partial^2 z}{\partial t^2} = f''(x+ct).(c)^2+\phi''(x-ct).(-c)^2$$

$$= c^2[f''(x+ct)+\phi''(x-ct)]$$

$$\therefore \frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2} \qquad [\because \text{ of (1)}]$$

Example 8. If $u = e^{ax}$ sin by, then prove that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$. (J.U. 2016)

Sol. Here $u = e^{ax}$ sin by ...(1)

Differentiating (1) partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(e^{ax}\sin by) = \sin by.\frac{\partial}{\partial x}(e^{ax})$$

$$= \sin by\,.\,a\,e^{ax} = a\sin by\,.\,e^{ax}$$

Now differentiating it partially w.r.t. y, we get

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}[a\sin by.e^{ax}] = ae^{ax}.\frac{\partial}{\partial y}(\sin by)$$

$$= a\,e^{ax}.b\cos by = a\,b\,e^{ax}\cos by$$

$$\therefore \frac{\partial^2 u}{\partial y \partial x} = a\,b\,e^{ax}\cos by \qquad ...(2)$$

Differentiating (1) partially w.r.t. y, we get

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(e^{ax}\sin by) = e^{ax}\frac{\partial}{\partial y}(\sin by) = e^{ax}.b\cos by$$

$$\therefore \frac{\partial u}{\partial y} = b\,e^{ax}\cos by$$

Now differentiating it partially w.r.t. x, we get

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(b\,e^{ax}\cos by) = b\cos by\,\frac{\partial}{\partial x}(e^{ax})$$

$$= b\cos by\,.\,a\,e^{ax}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\therefore \quad \frac{\partial^2 u}{\partial x \partial y} = a b e^{ax} \cos by \qquad \ldots(3)$$

From (2) and (3), we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 x}{\partial y \partial x}.$$

Example 9. If $x^x y^y z^z = c$ show that at $x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$.

Sol. $\because \quad x^x y^y z^z = c$

$$\because \qquad z^z = \frac{c}{x^x . y^y}$$

$$\Rightarrow \quad \log z^z = \log \left(\frac{c}{x^x . y^y} \right) \quad \Rightarrow \quad z \log z = \log c - \log x^x - \log y^y$$

$$\Rightarrow \quad z \log z = \log c - x \cdot \log x - y \cdot \log y$$

Differentiating partially w.r.t x,

$$\left(z . \frac{1}{z} + \log z . 1 \right) \frac{\partial z}{\partial x} = 0 - \left(x . \frac{1}{x} + \log x . 1 \right) - 0$$

$$\therefore \quad \frac{\partial z}{\partial x} = -\frac{1 + \log x}{1 + \log z} \qquad \ldots(1)$$

Similarly $\frac{\partial z}{\partial y} = -\frac{1 + \log y}{1 + \log z}$ $\qquad \ldots(2)$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left[-\frac{1 + \log y}{1 + \log z} \right] = -(1 + \log y) \frac{\partial}{\partial x} \left[\frac{1}{1 + \log z} \right]$$

$$= -(1 + \log y) \left[-(1 + \log z)^{-2} . \frac{1}{z} . \frac{\partial z}{\partial x} \right]$$

$$= -(1 + \log y) \left[\frac{-1}{(1 + \log z)^2} . \frac{1}{z} . \frac{-(1 + \log x)}{1 + \log z} \right] \qquad [\because \text{ of } (1)]$$

$$= -\left[\frac{(1 + \log x)(1 + \log y)}{z (1 + \log z)^3} \right]$$

At $x = y = z$, we have,

$$\frac{\partial^2 z}{\partial x \partial y} = -\left[\frac{(1 + \log x)(1 + \log x)}{x (1 + \log x)^3} \right]$$

$$= -\frac{1}{x (1 + \log x)} = -\frac{1}{x (\log e + \log x)} = -\frac{1}{x \log e x}$$

$$\therefore \quad \frac{\partial^2 z}{\partial x \partial y} = -(x \log e x)^{-1}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 10. If $u = \log\sqrt{x^2+y^2+z^2}$, prove that

$$(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=1.$$

Sol. Here $u = \log\sqrt{x^2+y^2+z^2} = \log\,(x^2+y^2+z^2)^{\frac{1}{2}}$

$$\therefore \qquad u = \frac{1}{2}\log(x^2+y^2+z^2)$$

$$\therefore \qquad \frac{\partial u}{\partial x} = \frac{1}{2(x^2+y^2+z^2)}.2x = \frac{x}{x^2+y^2+z^2}$$

and $$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2+z^2).1-x.2x}{(x^2+y^2+z^2)^2} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}$$

Similarly $$\frac{\partial^2 u}{\partial y^2} = \frac{z^2+x^2-y^2}{(x^2+y^2+z^2)^2}$$

and $$\frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}+\frac{z^2+x^2-y^2}{(x^2+y^2+z^2)^2}+\frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

$$= \frac{y^2+z^2-x^2+z^2+x^2-y^2+x^2+y^2-z^2}{(x^2+y^2+z^2)^2} = \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2}$$

$$\therefore \quad \frac{\partial^2 x}{\partial x^2}+\frac{\partial^2 x}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \frac{1}{x^2+y^2+z^2}$$

$$\therefore \quad (x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=1$$

Example 11. If $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u}=1$, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$

Sol. Here $$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1 \qquad ...(1)$$

or $$x^2(a^2+u)^{-1}+y^2(b^2+u)^{-1}+z^2(c^2+u)^{-1}=1$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Differentiating partially w.r.t. x, we have

$$2x(a^2+u)^{-1} - x^2(a^2+u)^{-2}\cdot\frac{\partial u}{\partial x} - y^2(b^2+u)^{-2}\cdot\frac{\partial u}{\partial x} - z^2(c^2+u)^{-2}\cdot\frac{\partial u}{\partial x} = 0$$

or $$\frac{2x}{a^2+u} = \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial x}$$

or $$\frac{2x}{a^2+u} = \mathrm{V}\frac{\partial u}{\partial x}$$

where $$\mathrm{V} = \frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2} \qquad \text{...(2)}$$

$\therefore$ $$\frac{\partial u}{\partial x} = \frac{2x}{\mathrm{V}(a^2+u)}$$

Similarly $$\frac{\partial u}{\partial y} = \frac{2y}{\mathrm{V}(b^2+u)}, \quad \frac{\partial u}{\partial z} = \frac{2z}{\mathrm{V}(c^2+u)}$$

L.H.S. $$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2$$

$$= \frac{4}{\mathrm{V}^2}\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]$$

$$= \frac{4}{\mathrm{V}^2}(\mathrm{V}) = \frac{4}{\mathrm{V}} \qquad [\because \text{ of (2)}]$$

R.H.S. $$= 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)$$

$$= 2\left[\frac{2x^2}{\mathrm{V}(a^2+u)} + \frac{2y^2}{\mathrm{V}(b^2+u)} + \frac{2z^2}{\mathrm{V}(c^2+u)}\right]$$

$$= \frac{4}{\mathrm{V}}\left[\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u}\right]$$

$$= \frac{4}{\mathrm{V}}(1) = \frac{4}{\mathrm{V}} \qquad [\because \text{ of (1)}]$$

$\therefore$ L.H.S. = R.H.S.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 12. (*a*) If $u = e^{xyz}$, show that $\dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z} = (1 + 3\,xyz + x^2y^2z^2)\,.\,e^{xyz}$

(*b*) If $z = x^y$, then prove that $\dfrac{\partial^3 z}{\partial^2 x\,\partial y} = \dfrac{\partial^3 z}{\partial x\,\partial y\,\partial x}$.

Sol. (*a*) Here $u = e^{xyz}$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}(e^{xyz}) = e^{xyz}.\frac{\partial}{\partial z}(xyz) = e^{xyz}.xy$$

$$\frac{\partial^2 u}{\partial y\,\partial z} = \frac{\partial}{\partial z}[xy\,e^{xyz}] = x\frac{\partial}{\partial y}[y\,e^{xyz}]$$

$$= x\left[y\,.\,(e^{xyz}.\,xz) + e^{xyz}\,.\,1\right] = (x^2yz + x)\,e^{xyz}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x\,\partial y\,\partial z} = \frac{\partial}{\partial x}\left[(x^2yx + x)\,e^{xyz}\right]$$

$$= (x^2yz + x)\frac{\partial}{\partial x}(e^{xyz}) + e^{xyz}\,\frac{\partial}{\partial x}(x^2yz + x)$$

$$= (x^2yx + x)\,.\,e^{xyz}\,.\,yz + e^{xyz}\,.\,(2\,xyz + 1)$$

$$= e^{xyz}\,[x^2y^2z^2 + xyz + 2\,xyz + 1]$$

$$\therefore \quad \frac{\partial^2 u}{\partial x\,\partial y\,\partial z} = (1 + 3\,xyz + x^2y^2z^2)\,e^{xyz}$$

(*b*) Here $z = x^y$

$$\therefore \quad \frac{\partial z}{\partial y} = x^y \log x$$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial x\,\partial y} = x^y.\frac{1}{x} + y\,x^{y-1}\log x = x^{y-1} + y\,x^{y-1}\log x$$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial x\,\partial y} = x^{y-1}\,(1 + y\log x)$$

$$\therefore \quad \frac{\partial^3 z}{\partial^2 x\,\partial y} = (y-1)\,x^{y-2}.(1 + y\log x) + \frac{y}{x}.x^{y-1}$$

$$= (y-1)x^{y-2}.(1 + y\log x) + y\,x^{y-2}$$

$$= x^{y-2}[(y-1)(1 + y\log x) + y]$$

$$\therefore \quad \frac{\partial^3 z}{\partial^2 x\,\partial y} = x^{y-2}\,[y + (y-1)(1 + y\log x)] \qquad \ldots(1)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Again $\dfrac{\partial z}{\partial x} = y x^{y-1}$

$$\therefore \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(y x^{y-1}) = y \,.\, x^{y-1} \log x + x^{y-1}.1$$

or $$\frac{\partial^2 z}{\partial y \partial x} = x^{y-1}\,(1 + y \log x)$$

$$\therefore \quad \frac{\partial^3 z}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\,[x^{y-1}(1+y\log x)]$$

$$= (y-1)\,x^{y-2}(1+y\log x) + x^{y-1}.\frac{y}{x}$$

$$= (y-1)\,x^{y-2}(1+y\log x) + y\,x^{y-2}$$

$$\therefore \quad \frac{\partial^3 z}{\partial x \partial y \partial x} = x^{y-2}[y+(y-1)(1+y\log x)] \qquad \ldots(2)$$

From (1) and (2), we get

$$\frac{\partial^3 z}{\partial^2 x \partial y} = \frac{\partial^3 z}{\partial x \partial y \partial x}.$$

Example 13. If $u = f(r)$ where $r^2 = x^2 + y^2$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f''(r) + \dfrac{1}{r} f'(r)$.

Sol. Here $r^2 = x^2 + y^2$ ...(1)

Differentiating (1) partially w.r.t. x,

$$2r\frac{\partial r}{\partial x} = 2x, \qquad \therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r} \qquad \ldots(2)$$

Differentiating (1) partially w.r.t. y,

$$2r\frac{\partial r}{\partial y} = 2y, \qquad \therefore \quad \frac{\partial r}{\partial y} = \frac{y}{r} \qquad \ldots(3)$$

Now $u = f(r)$

$$\therefore \quad \frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x} = \frac{x}{r} f'(r) \qquad [\because \text{ of } (1)]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{r\dfrac{\partial}{\partial x}[x f'(r)] - x f'(r)\dfrac{\partial}{\partial x}(r)}{r^2} = \frac{r\left[x f''(r).\dfrac{\partial r}{\partial x} + f'(r).1\right] - x f'(r)\dfrac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r\left[x f''(r).\dfrac{x}{r} + f'(r)\right] - \dfrac{x}{r}.x f'(r)}{r^2}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{x^2 f''(r) + r f'(r) - \frac{x^2}{r} f'(r)}{r^2} \quad \ldots(4)$$

$$\text{Similarly } \frac{\partial^2 u}{\partial y^2} = \frac{y^2 f''(r) + r f'(r) - \frac{y^2}{r} f'(r)}{r^2} \quad \ldots(5)$$

Adding (4) and (5), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2) f''(r) + 2r f'(r) - \frac{x^2 + y^2}{r} f'(r)}{r^2}$$

$$= \frac{r^2 f''(r) + 2r f'(r) - \frac{r^2}{r} f'(r)}{r^2} = \frac{r^2 f''(r) + r f'(r)}{r^2} = \frac{r^2 f''(r)}{r^2} + \frac{r f'(r)}{r^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$

Example 14. If $V = r^m$ where $r^2 = x^2 + y^2 + z^2$, show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = m(m+1)\, r^{m-2}$$

Sol. Here $r^2 = x^2 + y^2 + z^2$, $\quad \therefore \quad r = (x^2 + y^2 + z^2)^{1/2}$

$$\therefore \quad r^m = (x^2 + y^2 + z^2)^{\frac{m}{2}}$$

$$\therefore \quad V = (x^2 + y^2 + z^2)^{\frac{m}{2}}$$

$$\therefore \quad \frac{\partial V}{\partial x} = \frac{m}{2}(x^2 + y^2 + z^2)^{\frac{m}{2}-1} \cdot 2x = m x (x^2 + y^2 + z^2)^{\frac{m-2}{2}}$$

$$\frac{\partial^2 V}{\partial x^2} = m\left[x \cdot \frac{m-2}{2}(x^2 + y^2 + z^2)^{\frac{m-4}{2}} \cdot 2x + (x^2 + y^2 + z^2)^{\frac{m-2}{2}} \cdot 1 \right]$$

$$\frac{\partial^2 V}{\partial x^2} = m(m-2) x^2 (x^2 + y^2 + z^2)^{\frac{m-4}{2}} + m(x^2 + y^2 + z^2)^{\frac{m-2}{2}} \quad \ldots(1)$$

$$\text{Similarly } \frac{\partial^2 V}{\partial y^2} = m(m-2) y^2 (x^2 + y^2 + z^2)^{\frac{m-4}{2}} + m(x^2 + y^2 + z^2)^{\frac{m-2}{2}} \quad \ldots(2)$$

$$\frac{\partial^2 V}{\partial z^2} = m(m-2) z^2 (x^2 + y^2 + z^2)^{\frac{m-4}{2}} + m(x^2 + y^2 + z^2)^{\frac{m-2}{2}} \quad \ldots(3)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Adding (1), (2), (3), we get,

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=m(m-2)(x^2+y^2+z^2)^{\frac{m-4}{2}}(x^2+y^2+z^2)$$

$$+\,3m(x^2+y^2+z^2)^{\frac{m-2}{2}}$$

$$=m\,(m-2)\,r^{m-4}.\,r^2+3\,m\,r^{m-2}=m\,(m-2)\,r^{m-2}+3\,m\,r^{m-2}$$

$$=[m\,(m-2+3\,m]\,r^{m-2}=m\,[m-2+3]\,r^{m-2}$$

$$=m\,(m+1)\,r^{m-2}$$

$$\therefore\quad \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=m\,(m+1)\,r^{m-2}.$$

EXERCISE 5 (*a*)

1. If $f(x,y)=\dfrac{x-y}{x+y}$, find $\dfrac{\partial f}{\partial x}$ at $(2,-1)$ from definition and verify your answer by differentiation rules.

2. Let $f(x,y)=\sqrt{x^4+y^4+1}$. Evaluate $f_x(1,2)$ and $f_y(1,2)$.

3. (*i*) Let $f(x)=\sin^{-1}\dfrac{x}{y}$. Find $f_x(0,1)$ and $f_y(0,1)$.

   (*ii*) Let $f(x)=e^{\sin x}\log y$. Find $f_x\left(\dfrac{\pi}{4},e\right)$ and $f_y\left(\dfrac{\pi}{2},1\right)$

4. (*i*) Let $f(x,y)=x^3+y^3-3\,a\,x\,y$. Find $f_x(a,a)$ and $f_y(a,a)$

   (*ii*) Let $f:\mathbf{R}^2\to\mathbf{R}$ be defined by $f(x,y)=\sqrt{1-2xy+y^2}$.

   Evaluate $f_x(1,3)$ and $f_y(1,2)$.

5. Find the first order partial derivatives of

   (*i*) $\log(x^2+y^2)$ (*ii*) $x^3+y^3-3\,a\,x\,a\,y$

6. Find partial derivatives of first order when

   (*i*) $z=\sin(x^2y^2)$ (*ii*) $z=\sin^{-1}\left(\dfrac{x}{y}\right)$

7. (*i*) If $z=f\left(\dfrac{y}{x}\right)$, then $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=0$

   (*ii*) If $z=\sin^{-1}\left(\dfrac{x}{y}\right)+\tan^{-1}\dfrac{y}{x}$, then $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=0$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

(*iii*) If $f(x, y) = \sqrt{x^2 - y^2}\, \sin^{-1}\left(\frac{y}{x}\right)$, then prove that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f(x, y)$$

(*iv*) If $u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

8. If $u = \log\left(\frac{x^2 + y^2}{x + y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$.

9. (*i*) If $z = xyf\left(\frac{x}{y}\right)$, prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$.

(*ii*) If $z = xy \tan \frac{y}{x}$, prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$.

10. If $z(x + y) = x^2 + y^2$, then show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$.

11. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that

(*i*) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x + y + z}$ (*ii*) $\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)^2 = \frac{9}{(x + z + z)^2}$

(*iii*) $\left(\frac{\partial}{dx} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x + y + z)^2}$.

12. (*i*) If $u = x^2(y - z) + y^2(z - x) + z^2(x - y)$, then $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

(*ii*) If $u = x^3 + y^3 + z^3 + 3xyz$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u$.

(*iii*) If $V = (1 - 2xy + y^2)^{-1/2}$, prove that $x\frac{\partial V}{\partial x} - y\frac{\partial V}{\partial y} = y^2 V^3$.

13. If $u = \frac{1}{\sqrt{1 - 2xy + y^2}}$, show that $\frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left(y^2 \frac{\partial u}{\partial y}\right) = 0$.

14. Find the second order partial derivatives of

(*i*) e^{x-y} (*ii*) e^{xy} (*iii*) x^y

(*iv*) $\log(e^x + e^y)$ (*v*) e^{x^y}

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

15. Find the second order partial derivatives of

(*i*) $f(x, y) = \sin\left(\frac{y}{x}\right)$ (*ii*) $f(x, y) = x^2 \sin(x + y)$

16. (*a*) Verify that $\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 f}{\partial x\, \partial y}$ where $f = \log\left(\frac{x^2 + y^2}{x\, y}\right)$.

(*b*) If $u = \tan^{-1}\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)$, then prove that $\frac{\partial^2 u}{\partial x\, \partial y} = \frac{1}{(1+x^2+y^2)^{\frac{3}{2}}}$

17. Verify that $\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x}$ where z is

(*a*) (*i*) $a x^2 + 2 h x y + b y^2$ (*ii*) $a x^3 + 3 b x^2 y + 3 c x y^2 + d y^3$

(*iii*) $\log\left(\frac{x^3 + y^3}{x\, y}\right)$ (*iv*) $x^y + y^x$

(*v*) $\frac{x\, y}{\sqrt{1 + x^2 + y^2}}$ (*vi*) $3 x y - y^3 + (y^2 - 2 x)^{3/2}$

(*b*) (*i*) $f = \tan^{-1}\left(\frac{x}{y}\right)$ (*ii*) $f = x \sin y + y \sin x$

(*iii*) $f = x \tan y + y \tan x$

18. (*i*) If $z = \tan^{-1}\left(\frac{y}{x}\right)$, prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

(*ii*) If $z = x^2 \tan^{-1}\left(\frac{y}{x}\right)$, find $\frac{\partial^2 z}{\partial x\, \partial y}$ at (1, 1).

(*iii*) If $z = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$, prove that $\frac{\partial^2 z}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$, $x \neq 0$, $y \neq 0$

19. (*i*) If $z = \phi(y + a x) + \psi(y - a x)$, show that $\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$.

(*ii*) If $z = f(x + ay) + f(x - ay)$, then prove that $\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$

(*iii*) If $u = f(y + 2x) + g(y - 2x)$, then show that $\frac{\partial^2 u}{\partial x^2} = 4 \frac{\partial^2 u}{\partial y^2}$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

20. If $z = \tan(y + ax) + (y - ax)^{\frac{3}{2}}$, prove that $\dfrac{\partial^2 z}{\partial x^2} = a^2 \dfrac{\partial^2 z}{\partial y^2}$.

21. (*i*) If $z = \log\sqrt{x^2 + y^2}$, prove that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

(*ii*) If $f(x, y) = \log(x^2 + y^2) + \tan^{-1}\dfrac{y}{x}$, then prove that $f_{xx} + f_{yy} = 0$.

(*iii*) If $u = e^x(x\cos y - y\sin x)$, then show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

(*iv*) If $u = \log(\tan x + \tan y)$, then prove that $\sin 2x \cdot \dfrac{\partial u}{\partial x} + \sin 2y \cdot \dfrac{\partial u}{\partial y} = 2$

(*v*) If $z = \log(e^x + e^y)$, then show that $z_{xx} \cdot z_{yy} = (z_{xy})^2$

22. If $u = \log(x^2 + y^2 + z^2)$, prove that $x\dfrac{\partial^2 u}{\partial y\,\partial z} = y\dfrac{\partial^2 u}{\partial z\,\partial x} = z\dfrac{\partial^2 u}{\partial x\,\partial y}$.

23. If $u = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$; $x^2 + y^2 + z^2 \neq 0$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$

24. If $u = f(ax^2 + 2hxy + by^2)$, $v = \phi(ax^2 + 2hxy + by^2)$ show that

$$\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right).$$

25. If $u = \log(x^3 + y^3 - x^2 y - xy^2)$, prove that

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = -4(x + y)^{-2}.$$

26. Find the value of $\dfrac{1}{a^2}\dfrac{\partial^2 z}{\partial x^2} + \dfrac{1}{b^2}\dfrac{\partial^2 z}{\partial y^2}$ when $a^2x^2 + b^2y^2 - c^2z^2 = 0$.

27. If $z = (x + y) + (x + y)\phi\left(\dfrac{y}{x}\right)$, prove that $x\left(\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y\,\partial x}\right) = y\left(\dfrac{\partial^2 z}{\partial y^2} - \dfrac{\partial^2 z}{\partial x\,\partial y}\right)$

28. If $V = (x^2 + y^2 + z^2)^m$, show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial y^2} = 2m(2m + 1)(x^2 + y^2 + z^2)^{m-1}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

ANSWERS

2. $\frac{\sqrt{2}}{3}$, $\frac{8\sqrt{2}}{3}$ **3.** (*i*) 1, 0 (*ii*) $\frac{1}{\sqrt{2}} e^{\frac{1}{\sqrt{2}}}$, e

4. (*i*) 0, 0 (*ii*) $-\frac{3}{2}$, 1

5. (*i*) $\frac{2x}{x^2+y^2}$, $\frac{2y}{x^2+y^2}$ (*ii*) $3x^2 - 3ay$ (*iii*) $3y^2 - 3ax$

6. (*i*) $2xy^2 \cos(x^2y^2)$, $2yx^2 \cos(x^2y^2)$

(*ii*) $\frac{|y|}{y\sqrt{y^2-x2}}$, $-\frac{|x|}{|y|\sqrt{y^2-x^2}}$

14. (*i*) e^{x-y}, e^{x-y}, $-e^{x-y}$, $-e^{x-y}$

(*ii*) $y^2 e^{xy}$, $x^2 e^{xy}$, $(xy+1)e^{xy}$

(*iii*) $y(y-1)\, x^{y-2}$, $x^y (\log x)^2$, $x^{y-1}(1+y\log x)$

(*iv*) $\frac{e^{x+y}}{(e^x+e^y)^2}$, $\frac{e^{x+y}}{(e^x+e^y)^2}$, $-\frac{e^{x+y}}{(e^x+e^y)^2}$

(*v*) $yx^{y-2} \cdot e^{x^y} \cdot [yx^y + y - 1]$, $e^{x^y} x^y (1+x^y)(\log x)^2$,

$e^{x^y} \cdot x^{y-1}[1 + y\log x + x^y \cdot y \log x]$

15. (*i*) $\frac{y}{x^4}\left[2x\cos\left(\frac{y}{x}\right) - y\sin\frac{y}{x}\right]$, $-\frac{1}{x^2}\sin\frac{y}{x}$

$\frac{1}{x^3}\left[y\sin\frac{y}{x} - x\cos\frac{y}{x}\right]$, $\frac{1}{x^3}\left[y\sin\frac{y}{x} - x\cos\frac{y}{x}\right]$

(*ii*) $2\sin(x+y) + 4x\cos(x+y) - x^2\sin(x+y)$, $-x^2\sin(x+y)$,

$-x^2\sin(x+y) + 2x\cos(x+y)$, $2x\cos(x+y) - x^2\sin(x+y)$

18. (*ii*) 1 **26.** $\frac{1}{c^2 z}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Art-4. Change of Independent Variables

Whenever there is some ambiguity regarding the choice of independent variable, we use some notations as :

Consider the two relations $x = r\cos\theta, y = r\sin\theta$,

Connecting the four variables x, y, r and θ. Here each variable can be expressed in terms of two of the remaining three variables.

e.g., x can be expressed in terms of

(*i*) r, θ (*ii*) θ, y (*iii*) r, y.

Now if we are to find $\frac{\partial x}{\partial r}$, then it has no meaning in (*ii*) where as it has meaning in (*i*) and (*iii*).

Further $\frac{\partial x}{\partial r}$ in (*i*) keeping θ constant is not equal to $\frac{\partial x}{\partial r}$ in (*iii*) keeping y constant.

$\therefore$ it is necessary to distinguish between the two values of $\frac{\partial x}{\partial r}$ which can be done by denoting them as $\left(\frac{\partial x}{\partial r}\right)_\theta$ and $\left(\frac{\partial x}{\partial r}\right)_y$ respectively.

Thus $\left(\frac{\partial x}{\partial r}\right)_\theta$ = partial derivative of x w.r.t. r where r and θ are the independent variables.

$\left(\frac{\partial x}{\partial r}\right)_y$ = partial derivative of x w.r.t. r where r and y are the independent variables.

Important Notes : Sometimes we don't use these notations especially when it is clear which of these is to be used

e.g., if in a problem we are to find $\frac{\partial x}{\partial r}$ and $\frac{\partial x}{\partial \theta}$, then it is understood that r and θ are the independent variables.

ILLUSTRATIVE EXAMPLES

Example 1. If $x = r\cos\theta, y = r\sin\theta$, prove that

$$\frac{\partial^2\theta}{\partial x\,\partial y} = \frac{\partial^2}{\partial x^2}(\log r) = -\frac{\partial^2}{\partial y^2}(\log r) = -\frac{1}{r^2}\cos 2\theta$$

Sol. Given $x = r\cos\theta,\ y = r\sin\theta$

$\therefore$ on dividing, $\tan\theta = \frac{y}{x}$ ...(1)

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Squaring and adding, $r^2 = x^2 + y^2$ ...(2)

From (1), $\theta = \tan^{-1}\dfrac{y}{x}$

$$\therefore \quad \frac{\partial\theta}{\partial y} = \frac{1}{1+\left(\dfrac{y}{x}\right)^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2\theta}{\partial x\,\partial y} = \frac{(x^2+y^2).1 - x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{r^2\sin^2\theta - r^2\cos^2\theta}{(r^2)^2}$$

$$= -\frac{\cos^2\theta - \sin^2\theta}{r^2} = -\frac{1}{r^2}\cos 2\theta \qquad \text{...(3)}$$

From (2), $\log r^2 = \log(x^2+y^2)$

$\therefore \quad 2\log r = \log(x^2+y^2)$

$$\Rightarrow \quad \log r = \frac{1}{2}\log(x^2+y^2)$$

$$\therefore \quad \frac{\partial}{\partial x}(\log r) = \frac{1}{2}\cdot\frac{1}{x^2+y^2}.2x = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2(\log r)}{\partial x^2} = \frac{(x^2+y^2).1 - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = -\frac{1}{r^2}\cos 2\theta \qquad \text{...(4)}$$

[$\because$ of (3)]

$$\frac{\partial}{\partial y}(\log r) = \frac{1}{2}\cdot\frac{1}{x^2+y^2}.2y = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2}{\partial y^2}(\log r) = \frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} = -\left(\frac{y^2-x^2}{(x^2+y^2)}\right)$$

$$\Rightarrow \quad -\frac{\partial^2}{\partial y^2}(\log r) = \frac{y^2-x^2}{(x^2+y^2)^2} = -\frac{1}{r^2}\cos 2\theta \qquad \text{...(5)}$$

From (3), (4) and (5), we get,

$$\frac{\partial^2\theta}{\partial x\,\partial y} = \frac{\partial^2}{\partial x^2}(\log r) = -\frac{\partial^2}{\partial y^2}(\log r) = -\frac{1}{r^2}\cos 2\theta$$

EXERCISE 5 (*b*)

1. If $x = r\cos\theta, y = r\sin\theta$, evaluate

$$\left[\frac{\partial x}{\partial\theta}\right]_r, \left[\frac{\partial x}{\partial r}\right]_\theta, \left[\frac{\partial y}{\partial\theta}\right]_r, \left[\frac{\partial y}{\partial r}\right]_\theta, \left[\frac{\partial x}{\partial\theta}\right]_y, \left[\frac{\partial y}{\partial r}\right]_x \text{ where}$$

$\left[\dfrac{\partial x}{\partial\theta}\right]_r$ denotes partial of x w.r.t. θ when r is regarded a constant.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

2. (*a*) If $x = r\cos\theta, y = r\sin\theta$, prove that

(*i*) $\dfrac{\partial^2 r}{\partial x^2}+\dfrac{\partial^2 r}{\partial y^2}=\dfrac{1}{r}\left[\left(\dfrac{\partial r}{\partial x}\right)^2+\left(\dfrac{\partial r}{\partial y}\right)^2\right]$ (*ii*) $\dfrac{\partial^2 r}{\partial x^2}\times\dfrac{\partial^2 r}{\partial y^2}=\left(\dfrac{\partial^2 r}{\partial x\,\partial y}\right)^2$

(*iii*) $\left(\dfrac{\partial r}{\partial x}\right)^2+\left(\dfrac{\partial r}{\partial y}\right)^2=1$

(*b*) $\dfrac{\partial^2\theta}{\partial x^2}+\dfrac{\partial^2\theta}{\partial y^2}=0$ except when $x=0,\ y=0$.

ANSWERS

1. $-r\sin\theta$, $\cos\theta$, $r\cos\theta$, $\sin\theta$, $-y\,\text{cosec}^2\theta$, $\dfrac{r}{\sqrt{r^2-x^2}}$

Art-5. Homogeneous Functions

A function z is said to be homogeneous of x and y if it can be expressed in the form $x^n f\left(\dfrac{y}{x}\right)$. Here n is degree of the function.

$$\text{Consider } z=\frac{x^3-y^3}{x+y}=\frac{x^3\left(1-\dfrac{y^3}{x^3}\right)}{x\left(1+\dfrac{y}{x}\right)}=x^2 f\left(\frac{y}{x}\right)$$

Here z is a homogeneous function of degree (or order) 2.

Note. If $z = f(x, y)$ is a homogeneous function of degree n, then $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ are homogeneous functions of degree $n-1$.

Proof. $\because$ z is homogeneous function of degree n

$$\therefore\ z=x^n f\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial x}=x^n\frac{\partial}{\partial x}\left[f\left(\frac{y}{x}\right)\right]+f\left(\frac{y}{x}\right)\frac{\partial}{\partial x}(x^n)=x^n.f'\left(\frac{y}{x}\right).\left(-\frac{y}{x^2}\right)+f\left(\frac{y}{x}\right).nx^{n-1}$$

$$=-yx^{n-2}f'\left(\frac{y}{x}\right)+nx^{n-1}f\left(\frac{y}{x}\right)=x^{n-1}\left[nf\left(\frac{y}{x}\right)-\frac{y}{x}f'\left(\frac{y}{x}\right)\right]=x^{n-1}\phi\left(\frac{y}{x}\right)$$

where $\phi\left(\dfrac{y}{x}\right)=nf\left(\dfrac{y}{x}\right)-\dfrac{y}{x}f'\left(\dfrac{y}{x}\right)$

$\dfrac{\partial z}{\partial x}$ is a homogeneous function of degree $n-1$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Again $\dfrac{\partial z}{\partial y} = x^n . f'\left(\dfrac{y}{x}\right).\dfrac{1}{x} = x^{n-1} f'\left(\dfrac{y}{x}\right) = x^{n-1}\psi\left(\dfrac{y}{x}\right)$ where $\psi\left(\dfrac{y}{x}\right) = f'\left(\dfrac{y}{x}\right)$

$\dfrac{\partial z}{\partial y}$ is a homogeneous function of degree $n-1$.

Note In general, a real function $f(x_1, x_2, \ldots, x_m)$ of m independent real variables $x_1, x_2, \ldots, x_m$ is called a homogeneous function of degree n iff $f(x_1, x_2, \ldots x_m)$ can be expressed as

$$f(x_1, x_2, \ldots, x_m) = x_1^n \phi\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots, \frac{x_m}{x_1}\right)$$

where ϕ is a function of $\dfrac{x_2}{x_1}, \dfrac{x_3}{x_1}, \ldots, \dfrac{x_m}{x_1}$.

(Here we can take any variable x_i outside).

Art-6. Euler's Theorem of Homogeneous Functions

If z be a homogeneous function of x, y of order n, then $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = n\,z$ for all $x, y \in$ the domain of the function.

Proof. Since z is a homogeneous function of x, y of order n.

$$\therefore \quad z = x^n f\left(\frac{y}{x}\right) \qquad \ldots(1)$$

$$\therefore \quad \frac{\partial z}{\partial x} = n\,x^{n-1} f\left(\frac{y}{x}\right) + x^n . f'\left(\frac{y}{x}\right).\left(-\frac{y}{x^2}\right)$$

$$\text{or} \quad \frac{\partial z}{\partial x} = n\,x^{n-1} f\left(\frac{y}{x}\right) - y\,x^{n-2} f'\left(\frac{y}{x}\right)$$

Multiplying both sides by x, we get,

$$x\frac{\partial z}{\partial x} = n\,x^n f\left(\frac{y}{x}\right) - y\,x^{n-1} f'\left(\frac{y}{x}\right)$$

$$\text{or} \quad x\frac{\partial z}{\partial x} = n\,z - y\,x^{n-1} f'\left(\frac{y}{x}\right) \qquad \ldots(2)$$

[$\because$ of (1)]

Again from (1), we have,

$$\frac{\partial z}{\partial y} = x^n f'\left(\frac{y}{x}\right).\frac{1}{x} \quad \text{or} \quad \frac{\partial z}{\partial x} = x^{n-1} f'\left(\frac{y}{x}\right)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Multiplying both sides by y, we get,

$$y\frac{\partial z}{\partial x} = y\,x^{n-1} f'\left(\frac{y}{x}\right) \qquad ...(3)$$

Adding (2) and (3), we get,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n\,z$$

Cor. If z is homogeneous function of x and y of degree n, then prove that

(*i*) $x\dfrac{\partial^2 z}{\partial x^2} + y\dfrac{\partial^2 z}{\partial x\,\partial y} = (n-1)\dfrac{\partial z}{\partial x}$

(*ii*) $x\dfrac{\partial^2 z}{\partial y\,\partial x} + y\dfrac{\partial^2 z}{\partial y^2} = (n-1)\dfrac{\partial z}{\partial y}$

(*iii*) $x^2\dfrac{\partial^2 z}{\partial x^2} + 2\,x\,y\dfrac{\partial^2 z}{\partial x\,\partial y} + y^2\dfrac{\partial^2 z}{\partial y^2} = n\,(n-1)\,z.$

Proof : Given z is a homogeneous function of x and y of degree n

$\therefore$ by Euler's theorem, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n\,z \qquad ...(1)$$

(*i*) Differentiating (1) partially w.r.t. x, we get

$$x\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x}.1 + y\frac{\partial^2 z}{\partial x\,\partial y} = n\frac{\partial z}{\partial x}$$

$$\Rightarrow \quad x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x\,\partial y} = (n-1)\frac{\partial z}{\partial x} \qquad ...(2)$$

(*ii*) Differentiating (1) partially w.r.t. y, we get

$$x\frac{\partial^2 z}{\partial y\,\partial x} + y\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y}.1 = n\frac{\partial z}{\partial y}$$

$$\Rightarrow \quad x\frac{\partial^2 z}{\partial y\,\partial x} + y\frac{\partial^2 z}{\partial y^2} = (n-1)\frac{\partial z}{\partial y} \qquad ...(3)$$

(*iii*) Multiplying (2) by x, (3) by y and adding, we get

$$x^2\frac{\partial^2 z}{\partial x^2} + 2\,x\,y\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) \qquad \left(\because \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}\right)$$

$$= (n-1)\,n\,z$$

$$= n\,(n-1)\,z. \qquad [\because \text{ of (1)}]$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Note : Homogeneous Function of Three Variables

A function u of three variables x, y, z is said to be homogeneous of degree n, if it can be expressed as

$$u = x^n f\left(\frac{y}{x}, \frac{z}{x}\right) \text{ or } y^n f\left(\frac{x}{y}, \frac{z}{y}\right) \text{ or } z^n f\left(\frac{x}{z}, \frac{y}{z}\right)$$

Example. If $u = (x^2+y^2+z^2)^{-\frac{1}{2}}$, then show by Euler's theorem that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u$$

Sol. Here $u = (x^2+y^2+z^2)^{-\frac{1}{2}} = \left[x^2\left(1+\frac{y^2}{x^2}+\frac{z^2}{x^2}\right)\right]^{-\frac{1}{2}}$

$$= x^{-1}\left[1+\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2\right] = x^{-1} f\left(\frac{y}{x}, \frac{z}{x}\right)$$

$\therefore$ u is a homogeneous function of degree -1 in x, y, z.

$\therefore$ by Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = (-1)(u)$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u.$$

Art-7. Generalisation of Euler's Theorem on Homogeneous Functions

If z is a homogeneous function of m independent variables $x_1, x_2, \ldots x_m$ of degree n, then

$$x_1\frac{\partial z}{\partial x_1} + x_2\frac{\partial z}{\partial x_2} + \ldots + x_m\frac{\partial z}{\partial x_m} = nz$$

Proof. Since z is a homogeneous function of m independent variables $x_1, x_2, \ldots x_m$ of degree n,

$\therefore$ it can be expressed as

$$z = x_1^n f\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots, \frac{x_m}{x_1}\right)$$

$$= x_1^n f(u_2, u_3, \ldots u_m) \text{ where } u_i = \frac{x_i}{x_1} \qquad \ldots(1)$$

i.e., $u_2 = \frac{x_2}{x_1}, u_3 = \frac{x_3}{x_1}, \ldots, u_m = \frac{x_m}{x_1}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Differentiating (1) partially w.r.t. x_1, we get

$$\frac{\partial z}{\partial x_1} = x_1^{\,n}\left(\frac{\partial f}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_1} + \frac{\partial f}{\partial u_3}\frac{\partial u_3}{\partial x_1} + \ldots + \frac{\partial f}{\partial u_m}\frac{\partial u_m}{\partial x_1}\right) + n\,x_1^{\,n-1} f(u_2, u_3, \ldots, u_m)$$

$$= x_1^{\,n}\left[\frac{\partial f}{\partial u_2}\left(-\frac{x_2}{x_1^{\,2}}\right) + \frac{\partial f}{\partial u_3}\left(-\frac{x_3}{x_1^{\,2}}\right) + \ldots + \frac{\partial f}{\partial u_m}\left(-\frac{x_m}{x_1^{\,2}}\right)\right]$$

$$+\; n\,x_1^{\,n-1} f(u_2, u_3, \ldots, u_m)$$

$$= n\,x_1^{\,n-1} f(u_2, u_3, \ldots, u_m) - x_1^{\,n-2}\left[x_2\frac{\partial f}{\partial u_2} + x_3\frac{\partial f}{\partial u_3} + \ldots + x_m\frac{\partial f}{\partial u_m}\right] \qquad \ldots(2)$$

Differentiating (1) partially w.r.t. x_2, we get,

$$\frac{\partial z}{\partial x_2} = x_1^{\,n}\frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial x_2} = x_1^{\,n}\frac{\partial f}{\partial u_2}\left(\frac{1}{x_1}\right) = x_1^{\,n-1}\frac{\partial f}{\partial u_2} \qquad \ldots(3)$$

Similarly we have

$$\frac{\partial z}{\partial x_3} = x_1^{\,n-1}\frac{\partial f}{\partial u_3} \qquad \ldots(4)$$

.....

.....

$$\frac{\partial z}{\partial x_m} = x_1^{\,n-1}\frac{\partial f}{\partial u_m} \qquad \ldots(m+1)$$

Multiplying equation (2), (3), (4),....., $(m + 1)$ by $x_1, x_2, \ldots, x_m$ respectively and adding, we get

$$x_1\frac{\partial z}{\partial x_1} + x_2\frac{\partial z}{\partial x_2} + \ldots + x_m\frac{\partial z}{\partial x_m} = n\,x_1^{\,n} f(u_2, u_3, \ldots u_m) = n\,z \qquad [\because \text{ of } (1)]$$

ILLUSTRATIVE EXAMPLES

Example 1. Verify Euler's Theorem for the function

$$f(x, y) = \frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} + y^{\frac{1}{2}}}$$

Sol. Here $f(x, y) = \dfrac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} + y^{\frac{1}{2}}} = z$ (say)

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\therefore \quad z = \frac{x^{\frac{1}{3}}\left[1+\left(\frac{y}{x}\right)^{\frac{1}{3}}\right]}{x^{\frac{1}{2}}\left[1+\left(\frac{y}{x}\right)^{\frac{1}{2}}\right]} = x^{\frac{1}{3}-\frac{1}{2}} f\left(\frac{y}{x}\right) = x^{-\frac{1}{6}} f\left(\frac{y}{x}\right).$$

$\therefore$ z is a homogeneous function of degree $-\frac{1}{6}$ in x and y.

Now to verify Euler's theorem, we are to show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -\frac{1}{6}z \qquad \ldots(1)$$

Now $$z = \frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}} \Rightarrow \log z = \log\left(\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}\right)$$

or $$\log z = \log\left(x^{\frac{1}{3}}+y^{\frac{1}{3}}\right) - \log\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right) \qquad \ldots(2)$$

Differentiating (2) partially w.r.t. x, we get

$$\frac{1}{z}\frac{\partial z}{\partial x} = \frac{\frac{1}{3}x^{-\frac{2}{3}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}} - \frac{\frac{1}{2}x^{-\frac{1}{2}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}$$

$$\therefore \quad \frac{1}{z}.x\frac{\partial z}{\partial x} = \frac{\frac{1}{3}x^{\frac{1}{3}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}} - \frac{\frac{1}{2}x^{\frac{1}{2}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}} \qquad \ldots(3)$$

Differentiating (2) partially w.r.t. y, we get

$$\frac{1}{z}\frac{\partial z}{\partial y} = \frac{\frac{1}{3}y^{-\frac{2}{3}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}} - \frac{\frac{1}{2}y^{-\frac{1}{2}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}$$

$$\therefore \quad \frac{1}{z}.y\frac{\partial z}{\partial y} = \frac{\frac{1}{3}y^{\frac{1}{3}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}} - \frac{\frac{1}{2}y^{\frac{1}{2}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}} \qquad \ldots(4)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Adding (3) and (4), we get

$$\frac{1}{z}\left[x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right]=\frac{1}{3}\cdot\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}-\frac{1}{2}\cdot\frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}$$

or $$\frac{1}{z}\left[x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right]=\frac{1}{3}-\frac{1}{2}$$

$\therefore$ $$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=-\frac{1}{6}z$$

which verifies Euler's theorem for the function

$$z=\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}.$$

Example 2. Verify Euler's Theorem for $z=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)$.

Sol. Here $z=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)$

$\therefore$ $$z=x^0\left(\text{some function of }\frac{y}{x}\right)$$

$\therefore$ z is a homogeneous function of degree 0.

Now to verify Euler's Theorem, we are to show that

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0 \qquad \text{(Proved earlier)}$$

Example 3. If $z=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\sin 2z$.

Sol. Let $$u=\frac{x^3+y^3}{x-y}=\frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left[1-\left(\frac{y}{x}\right)\right]}=x^2 f\left(\frac{y}{x}\right)$$

$\therefore$ u is a homogeneous function of degree 2.

$\therefore$ by Euler's Theorem, $$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2u \qquad ...(1)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Now $\quad u = \dfrac{x^3 + y^3}{x - y}$

$\therefore \quad z = \tan^{-1} u$ or $u \tan z$

$\therefore \quad \dfrac{\partial u}{\partial x} = \sec^2 z \dfrac{\partial z}{\partial x}, \dfrac{\partial u}{\partial y} = \sec^2 z \dfrac{\partial z}{\partial y}$

$\therefore$ from (1), we get, $x \sec^2 z \cdot \dfrac{\partial z}{\partial x} + y \sec^2 z \dfrac{\partial z}{\partial y} = 2 \tan z$

$$\Rightarrow \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2\,\frac{\tan z}{\sec^2 z} \quad \Rightarrow \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \sin 2z$$

$$\left[\because \frac{2\tan z}{\sec^2 z} = \frac{\dfrac{2\sin z}{\cos z}}{\dfrac{1}{\cos^2 z}} = 2\sin z\cos z = \sin 2z\right]$$

Example 4. If $\sin z' = \dfrac{x + 2y + 3z}{\sqrt{x^8 + y^8 + z^8}}$, prove that $x\dfrac{\partial z'}{\partial x} + y\dfrac{\partial z'}{\partial y} + z\dfrac{\partial z'}{\partial z} + 3\tan z' = 0.$

Sol. We have

$$\sin z' = \frac{x + 2y + 3z}{\sqrt{x^8 + y^8 + z^8}} = \frac{x\left[1 + \dfrac{2y}{x} + \dfrac{3z}{x}\right]}{x^4\sqrt{1 + \left(\dfrac{y}{x}\right)^8 + \left(\dfrac{z}{x}\right)^8}} = x^{-3} f\left(\frac{y}{x}, \frac{z}{x}\right)$$

$\therefore$ $\sin z'$ is a homogeneous function of degree -3 in x, y, z.

$\therefore$ by Euler's Theorem, we have

$$x\frac{\partial}{\partial x}(\sin z') + y\frac{\partial}{\partial y}(\sin z') + z\frac{\partial}{\partial z}(\sin z') = -3\sin z'$$

or $\quad x\cos z'\dfrac{\partial z'}{\partial x} + y\cos z'\dfrac{\partial z'}{\partial y} + z\cos z'\dfrac{\partial z'}{\partial z} = -3\sin z'$

Dividing both sides by $\cos u'$, we get

$$x\frac{\partial z'}{\partial x} + y\frac{\partial z'}{\partial y} + z\frac{\partial z'}{\partial z} = -3\tan z'$$

$$\therefore \quad x\frac{\partial z'}{\partial x} + y\frac{\partial z'}{\partial y} + z\frac{\partial z'}{\partial z} + 3\tan z' = 0.$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 5. If $u = x\, f\left(\frac{y}{x}\right) + y\,\phi\left(\frac{y}{x}\right)$, prove that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0$

Sol. Here $u = x\, f\left(\frac{y}{x}\right) + y\,\phi\left(\frac{y}{x}\right)$

$$\therefore \quad u = x\left\{f\left(\frac{y}{x}\right) + \frac{y}{x}.\phi\left(\frac{y}{x}\right)\right\}$$

$\Rightarrow$ u is a homogeneous function of degree 1

$\therefore$ by Euler's Theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$ ...(1)

Differentiating (1) partially w.r.t. x,

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial u}{\partial x}$$

or $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = 0$ ...(2)

Differentiating (1) partially w.r.t. y, $x\frac{\partial^2 u}{\partial y\,\partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \frac{\partial u}{\partial y}$

or $x\frac{\partial^2 u}{\partial y\,\partial x} + y\frac{\partial^2 u}{\partial y^2} = 0$ ...(3)

Multiplying (2) by x, (3) by y and adding,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$$

Example 6. If $z = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \tan z$

and $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = \tan^3 z$.

Sol. $z = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right) \Rightarrow \sin z = \frac{x^2+y^2}{x+y} = u$ (say)

$\therefore$ u *i.e.* $\sin z$ is a homogeneous function of degree 1.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$\therefore$ by Euler's Theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u \qquad ...(1)$$

Now $u=\sin z$

$$\therefore \quad \frac{\partial u}{\partial x}=\cos z\,.\,\frac{\partial z}{\partial x},\ \frac{\partial u}{\partial y}=\cos z\,.\,\frac{\partial z}{\partial y}$$

$\therefore$ from (1), we have

$$x\cos z\,.\,\frac{\partial z}{\partial x}+y\cos z\,.\,\frac{\partial z}{\partial y}=\sin z$$

Dividing both sides by $\cos z$, we have

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\tan z \qquad ...(2)$$

Differentiating (2) partially w.r.t. x, we get

$$x\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}+y\frac{\partial^2 z}{\partial x\,\partial y}=\sec^2 z\,.\,\frac{\partial z}{\partial x} \qquad ...(3)$$

Differentiating (2) partially w.r.t. y, we get

$$x\frac{\partial^2 z}{\partial x\,\partial y}+y\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial y}=\sec^2 z\,.\,\frac{\partial z}{\partial y} \qquad ...(4)$$

Multiplying (3) by x and (4) by y, we get

$$x^2\frac{\partial^2 z}{\partial x^2}+x\frac{\partial z}{\partial x}+xy\frac{\partial^2 z}{\partial x\,\partial y}=\sec^2 z\,.\,x\frac{\partial z}{\partial x} \qquad ...(5)$$

$$xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2}+y\frac{\partial z}{\partial y}=\sec^2 z\,.\,y\frac{\partial z}{\partial y} \qquad ...(6)$$

Adding (5) and (6), we get

$$x^2\frac{\partial^2 z}{\partial x^2}+y^2\frac{\partial^2 z}{\partial y^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=\sec^2 z\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)$$

$$\therefore \quad x^2\frac{\partial^2 z}{\partial x^2}+y^2\frac{\partial^2 z}{\partial y^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+\tan z=(1+\tan^2 z)\,.\,\tan z \qquad [\because \text{ of } (2)]$$

$$\therefore \quad x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=\tan^3 z\,.$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

EXERCISE 5 (c)

1. Verify Euler's Theorem for

(a) (i) $z = a x^2 + 2 h x y + b y^2$ (ii) $z = x^n \log \dfrac{y}{x}$

(iii) $z = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$ (iv) $z = \dfrac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{4}} + y^{\frac{1}{4}}}$

(v) $z = (x^2 + x y + y^2)^{-1}$ (vi) $z = \dfrac{x + y}{x^2 + y^2}$.

(b) (i) $u = \sqrt{x^2 + y^2}$ (ii) $u = xy + yz + zx$

(iii) $u = x^3 - 3x^2 y - y^3$ (iv) $u = e^{\frac{-x}{y}}$

(v) $z = \dfrac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x^n + y^n}$ (vi) $z = x^n \sin \dfrac{y}{x}$

2. Verify Euler's Theorem for

(i) $z = x^4 \log \dfrac{y}{x}$ (ii) $z = \dfrac{x y}{x + y}$

(iii) $z = x^2 e^{-y/x}$ (iv) $z = x y f\left(\dfrac{x}{y}\right)$

3. Show that the following functions are homogeneous, find their degree & verify Euler's Theorem :

(i) $z = y e^{-x/y}$ (ii) $f(x, y) = 4 x^4 + 8 x^2 y^2 + 8 y^4$

(iii) $u = \dfrac{1}{x^2 + y^2 + z^2}$ (iv) $u = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$

4. (i) If H is a homogeneous function of x, y, z of order n, prove that

$$x \frac{\partial H}{\partial x} + y \frac{\partial H}{\partial y} + z \frac{\partial H}{\partial z} = nH.$$

(ii) State and prove Euler's Theorem on Homogeneous functions of three variables.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

5. (*i*) If $u = \tan^{-1}\left(\dfrac{x^2 - y^2}{x + y}\right)$, then prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\sin 2u$

(*ii*) If $u = \sec^{-1}\dfrac{x^3 - y^3}{x + y}$, then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2\cot u$

6. (*i*) If $u = \sin^{-1}\left(\dfrac{x^2 + y^2}{x + y}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$.

(*ii*) If $z = \sin^{-1}\left(\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right)$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \dfrac{1}{2}\tan z$.

(*iii*) If $z = \sin^{-1}\left(\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right)$, show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0 \qquad \text{or} \qquad \frac{\partial z}{\partial x} = -\frac{y}{x}\frac{\partial z}{\partial y}.$$

(*iv*) If $z = \cos^{-1}\left(\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right)$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = -\dfrac{1}{2}\cot z$

(J.U. 2016)

(*v*) If $z = \tan^{-1}\left(\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right)$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \dfrac{1}{4}\sin 2z$

(*vi*) If $z = \log\left(\dfrac{x^2 + y^2}{x + y}\right)$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 1$

(*vii*) If $u = \log\left(\dfrac{x^4 + y^4}{x + y}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial x}{\partial y} = 3$

(*viii*) If $u = \log\dfrac{x^4 - y^4}{x - y}$, then prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3$

7. If $u = \log\left(\dfrac{x^5 + y^5 + z^5}{xy + yz + zx}\right)$, then prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 3$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

8. (*i*) If $u = \dfrac{x^2 y^2}{x+y}$, prove that $x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial x \partial y} = 2\dfrac{\partial u}{\partial x}$

(*ii*) If $u = \dfrac{x^2 y^2}{x+y}$, then show that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 6u$

(*iii*) If $u = \dfrac{x^2 y^2}{x^2 + y^2}$, prove that $x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial u}{\partial x}$

(*iv*) If $u = \dfrac{xy}{\sqrt{x}+\sqrt{y}}$, then show that $x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{1}{2}\dfrac{\partial u}{\partial x}$

9. If $u = x^2 \tan^{-1}\dfrac{y}{x} - y^2 \tan^{-1}\dfrac{x}{y}$, then evaluate

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}.$$

10. If $z = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$, prove that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (1 - 4\sin^2 z)\sin 2z.$$

11. If $u = x\left[\phi\left(\dfrac{y}{x}\right) + \psi\left(\dfrac{y}{x}\right)\right]$, then show that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$

12. If $u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{-\sin u \cos 2u}{4\cos^3 u}$$

13. If $u = \sin^{-1}\left[\dfrac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} + y^{\frac{1}{2}}}\right]^{\frac{1}{2}}$, using Euler's theorem find the value of

$$x^2\frac{\partial^2 x}{\partial x^2} + 2xy\frac{\partial^2 x}{\partial x \partial y} + y^2\frac{\partial^2 x}{\partial y^2}.$$

ANSWERS

9. $2u$

13. $\dfrac{\tan u}{144}(13 + \tan^2 u)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

6

TANGENTS AND NORMAL

Art-1. Slope of Tangent to a Curve

We know that $\frac{dy}{dx}$ at any point (x, y) of a curve $y = f(x)$ is the tangent of the angle which the tangent line at P makes with the positive direction of x-axis.

$\therefore$ $\frac{dy}{dx}$ at any point (x, y) of any curve $y = f(x)$ is the slope (or gradient) of the tangent to the curve at the point (x, y).

Note. 1. If ψ is the angle which the tangent to the curve makes with x-axis, then $\frac{dy}{dx} = \tan\psi$.

2. If the tangent is parallel to x-axis, then $\psi = 0$

$\therefore$ $\tan\psi = \tan 0 = 0$ $\qquad$ $\therefore$ $\frac{dy}{dx} = 0$

3. If the tangent is parallel to y-axis, then $\psi = 90°$

$\therefore$ $\cot\psi = \cot 90° = 0$

$\therefore$ $\frac{1}{\frac{dy}{dx}} = 0$ $\qquad$ or $\frac{dx}{dy} = 0$

ILLUSTRATIVE EXAMPLES

Example 1. Prove that the tangents to the curve $y = x^2 - 5x + 6$ at the point (2, 0) and (3, 0) are at right angles.

Sol. The equation of the curve is

$$y = x^2 - 5x + 6$$

$$\therefore \quad \frac{dy}{dx} = 2x - 5$$

Let m_1, m_2 be slopes of tangents to the curve at the points (2, 0), (3, 0)

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$\therefore \quad m_1 = \text{value of } \dfrac{dy}{dx} \text{ at } (2, 0) = 2(2) - 5 = 4 - 5 = -1$

$m_2 = \text{value of } \dfrac{dy}{dx} \text{ at } (3, 0) = 2(3) - 5 = 6 - 5 = 1$

$\therefore \quad m_1 m_2 = (-1)(1) = -1$

$\therefore$ tangents to the given curve at (2, 0), (3, 0) are perpendicular to each other.

Example 2. At what point of the curve $y = x^2$ does the tangent make an angle of 45° with x-axis ?

Sol. The equation of the curve is $y = x^2$

$$\therefore \quad \frac{dy}{dx} = 2x$$

At (x, y), slope of tangent $= 2x$

$\because$ tangent makes an angle of 45° with x-axis

$\therefore$ slope of tangent = tan 45° = 1

$$\therefore \quad 2x = 1 \quad \Rightarrow \quad x = \frac{1}{2}$$

When $x = \dfrac{1}{2}$, $y = \left(\dfrac{1}{2}\right)^2 = \dfrac{1}{4}$

$\therefore$ point is $\left(\dfrac{1}{2}, \dfrac{1}{4}\right)$

Example 3. Find the slope of the normal at the point (am^3, am^2) to the curve $ax^2 = y^3$.

Sol. The equation of the curve is

$$ax^2 = y^3$$

Differentiating both sides w.r.t. x, we get,

$$2ax = 3y^2 \frac{dy}{dx}$$

$$\therefore \quad \frac{dy}{dx} = \frac{2ax}{3y^2}$$

At (am^3, am^2), $\dfrac{dy}{dx} = \dfrac{2a(am^3)}{3(am^2)^2} = \dfrac{2a^2m^3}{3a^2m^4} = \dfrac{2}{3m}$

$\therefore$ at (am^3, am^2), slope of tangent $= \dfrac{2}{3m}$

$\therefore$ at (am^3, am^2), slope of normal $= -\dfrac{1}{\frac{2}{3m}} = -\dfrac{3m}{2}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 4. Find the point on the curve

$$x = a(\theta + \sin\theta), \quad y = a(1 - \cos\theta)$$

where tangent is parallel to x-axis.

Sol. The equations of curve are

$$x = a(\theta + \sin\theta), \; y = a(1 - \cos\theta)$$

$$\therefore \quad \frac{dx}{d\theta} = a(1 + \cos\theta), \quad \frac{dy}{d\theta} = a\sin\theta$$

$$\text{Now} \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

Since tangent is perpendicular to x-axis

$$\therefore \quad \frac{dx}{dy} = 0 \qquad \Rightarrow \cot\frac{\theta}{2} = 0$$

$$\Rightarrow \quad \frac{\theta}{2} = \frac{\pi}{2} \quad \Rightarrow \theta = \pi$$

$$\therefore \quad x = a(\pi + \sin\pi) = a(\pi + 0) = a\pi$$

$$y = a(1 - \cos\pi) = a(1 + 1) = 2a$$

$\therefore$ points is $(a\pi, 2a)$

EXERCISE 6 (a)

1. Prove that the tangents to the curve $y = x^2 - 3x + 4$ at (1, 0) and (2, 2) are perpendicular to each other.
2. Show that the tangents to the curve $y = 2x^3 - 3$ at the points when $x = 2$ and $x = -2$ are parallel.
3. Find the slope of the tangent to the curve

   (*i*) $x = at^2, \; y = 2at$ at $t = 1$

   (*ii*) $x = a\cos\theta, \; y = a\sin\theta$ at $\theta = \frac{\pi}{4}$
4. Find the slope of the normal to the curve $y = 3x^2$ at the point whose x-coordinate is 2.
5. Find a point on the parabola $y = (x - 3)^2$ where the tangents is parallel to the line joining (3, 0) and (4, 1).

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

6. Find the points on the curve $y = 12x - x^3$, where the tangent is parallel to x-axis.

7. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangent is parallel to the y-axis.

ANSWERS

3. (*i*) 1 (*ii*) -1

4. $-\frac{1}{12}$ 5. $\left(\frac{7}{2}, \frac{1}{4}\right)$

6. (2, 16), (–2, –16) 7. (–1, 0), (3, 0)

Art-2. Equation of Tangent and Normal

Let $y = f(x)$ be a curve and let (x_0, y_0) be a point on the curve.

(*i*) The equation of tangent at (x_0, y_0) is

$$y - y_0 = f'(x_0)(x - x_0)$$

(*ii*) The equation of normal at (x_0, y_0) is

$$(y - y_0) f'(x_0) + (x - x_0) = 0$$

Proof. The equation of curve is $y = f(x)$

We know that $\frac{dy}{dx} = f'(x)$, if it exists, is the slope of the tangent to the curve at (x, y)

$\therefore$ $f'(x_0)$ is the slope of tangent to the curve at (x_0, y_0)

$\therefore$ the equation of tangent to the curve at (x_0, y_0) is

$$y - y_0 = f'(x_0)(x - x_0)$$

Also slope of tangent at $(x_0, y_0) = f'(x_0)$

$\therefore$ slope of normal at $(x_0, y_0) = -\frac{1}{f'(x_0)}$, provided $f'(x_0) \neq 0$

$\therefore$ the equation of normal to the curve at (x_0, y_0) is

$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$$

or $(y - y_0) f'(x_0) = -(x - x_0)$

or $(y - y_0) f'(x_0) + (x - x_0) = 0$

In case $f'(x_0) = 0$, the tangent is parallel to x-axis and therefore, normal is parallel to y-axis.

$\therefore$ the equation of normal is

$$x = x_0 \quad \text{or} \quad x - x_0 = 0$$

This can be written as

$$(y - y_0) f'(x_0) + (x - x_0) = 0$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Note 1. Equation of tangent at (x_1, y_1)

Let (x_1, y_1) be any point on the curve and

$$m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$ be the slope of tangent at (x_1, y_1)

$\therefore$ equation of tangent at (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

Note 2. Equation of normal at (x_1, y_1)

Slope of tangent at $(x_1, y_1) = m$

$\therefore$ slope of normal at $(x_1, y_1) = -\dfrac{1}{m}$

$\therefore$ equation of normal at (x_1, y_1) is

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

Note : When any line is parallel to y-axis, then its equation through (x_1, y_1) is $x = x_1$.

Art-3. Find the equation of the tangent and normal at a point t_1 to the curve $x = f(t), y = \phi(t)$.

Proof : The equations of curve are

$$x = f(t),\ y = \phi(t)$$

$$\therefore \quad \frac{dx}{dt} = f'(t),\ \frac{dy}{dt} = \phi'(t)$$

Now $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\phi'(t)}{f'(t)}$$

At $t = t_1$, $\dfrac{dy}{dx} = \dfrac{\phi'(t_1)}{f'(t_1)}$, which is slope of tangent.

$\therefore$ equation of tangent at $t = t_1$ is

$$y - \phi(t_1) = \frac{\phi'(t_1)}{f'(t_1)}[x - f(t_1)]$$

Slope of normal $= -\dfrac{1}{\text{slope of tangent}} = -\dfrac{f'(t_1)}{\phi'(t_1)}$

$\therefore$ equation of normal at $t = t_1$ is

$$y - \phi(t_1) = -\frac{f'(t_1)}{\phi'(t_1)}[x - f t_1)]$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

ILLUSTRATIVE EXAMPLES

Example 1. Find the equations of the tangent and normal to the curve $y = x^2 + 4x + 1$ at the point whose abscissa is 3.

Sol. The equation of curve is

$$y = x^2 + 4x + 1 \quad ...(1)$$

When $x = 3$, $y = 9 + 12 + 1 = 22$

$\therefore$ point with abscissa 3 is (3, 22)

Diff. (1) w.r.t.x, $\dfrac{dy}{dx} = 2x + 4$

At (3, 22), $\dfrac{dy}{dx} = 6 + 4 = 10$

$\therefore$ slope of tangent = 10

$\therefore$ equation of tangent at (3, 22) is

$$y - 22 = 10\,(x - 3)$$

or $y - 22 = 10x - 30$ or $10x - y - 8 = 0$

Also slope of normal $= -\dfrac{1}{10}$

$\therefore$ equation of normal at (3, 22) is

$$y - 22 = -\frac{1}{10}\,(x - 3) \quad \text{or} \quad 10\,y - 220 = -x + 3$$

or $x + 10\,y - 223 = 0$

Example 2. Find the equation of the tangent line to the curve $y = x^2 + 4\,x - 16$ which is parallel to the line $6\,x - 2\,y - 3 = 0$.

Sol. The equation of curve is

$$y = x^2 + 4x - 16 \quad ...(1)$$

Diff. w.r.t x, $\dfrac{dy}{dx} = 2x + 4$

$\therefore$ at (x, y), slope of tangent $= 2x + 4$

$\therefore$ $m_1 = 2x + 4$, where m_1 is slope of tangent.

Let m_2 be the slope of line $6\,x - 2\,y - 3 = 0$ ...(2)

$\therefore$ $m_2 = -\dfrac{6}{-2} = 3$

$\because$ tangent is parallel to line (2)

$\therefore$ $m_1 = m_2$

$\therefore$ $2x + 4 = 3$, $\therefore$ $x = -\dfrac{1}{2}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Putting $x = -\frac{1}{2}$ in (1), $y = \frac{1}{4} - 2 - 16 = -\frac{71}{4}$

$\therefore$ points is $\left(-\frac{1}{2}, -\frac{71}{4}\right)$

Also slope of tangent = 3

$\therefore$ equation of tangent at $\left(-\frac{1}{2}, -\frac{71}{4}\right)$ is

$$y - \left(-\frac{71}{4}\right) = 3\left[x - \left(-\frac{1}{2}\right)\right]$$

or $y + \frac{71}{4} = 3x + \frac{3}{2}$ or $4y + 71 = 12x + 6$

or $12x - 4y - 65 = 0$

Example 3. Find the equation of the tangent and normal at $\theta = \frac{\pi}{2}$ to the cycloid

$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$.

Sol. The equations of cycloid are

$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$$

$$\therefore \quad \frac{dx}{d\theta} = a(1 - \cos\theta),\ \frac{dy}{d\theta} = a\sin\theta$$

Now $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

At $\theta = \frac{\pi}{2}$, $\frac{dy}{dx} = \frac{\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}} = \cot\frac{\pi}{4} = 1$, which is slope of tangent

When $\theta = \frac{\pi}{2}$,

$$x = a\left(\frac{\pi}{2} - \sin\frac{\pi}{2}\right) = a\left(\frac{\pi}{2} - 1\right)$$

$$y = a\left(1 - \cos\frac{\pi}{2}\right) = a(1 - 0) = a$$

$\therefore$ point is $\left(a\left(\frac{\pi}{2} - 1\right), a\right)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

The equation of tangent at $\left(a\left(\frac{\pi}{2}-1\right), a\right)$ is

$$y-a=1\left[x-a\left(\frac{\pi}{2}-1\right)\right]$$

or $\quad y-a=x-\frac{a\pi}{2}+a$

or $\quad x-y=\frac{a\pi}{2}-2a$

Again, slope of normal $=-\frac{1}{1}=-1$

$\therefore$ equation of normal is

$$y-a=-1\left[x-a\left(\frac{\pi}{2}-1\right)\right]$$

or $\quad y-a=-x+\frac{a\pi}{2}-a$

or $\quad x+y=\frac{a\pi}{2}$

Example 4. Prove that $\frac{x}{a}+\frac{y}{b}=1$ touches the curve $y=b\,e^{-\frac{x}{a}}$ at the point where the curve crosses the axis of y. Also write down the equation of the normal at the same point.

Sol. The equation of curve is

$$y=b\,e^{-\frac{x}{a}} \qquad ...(1)$$

It crosses y-axis where $x=0$

$\therefore$ putting $x=0$ in (1), we get,

$y=b\,e^0$ or $y=b$

$\therefore$ curve (1) crosses y-axis at $(0, b)$.

From (1), $\frac{dy}{dx}=-\frac{b}{a}e^{-\frac{x}{a}}$

At $(0, b)$, $\frac{dy}{dx}=-\frac{b}{a}e^0=-\frac{b}{a}$, which is slope of tangent

$\therefore$ equation of tangent at $(0, b)$ is

$$y-b=-\frac{b}{a}(x-0)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

or $\frac{y}{b} - 1 = -\frac{x}{a}$

or $\frac{x}{a} + \frac{y}{b} = 1$

which is same as the equation of given line.

Hence the result.

Slope of normal at $(0, b) = \frac{a}{b}$

$\therefore$ equation of normal at $(0, b)$ is

$$y - b = \frac{a}{b}(x - 0)$$

or $by - b^2 = ax$

or $ax - by + b^2 = 0.$

Example 5. Prove that $\frac{x}{a} + \frac{y}{b} = 2$ touches the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ at the point (a, b) for all values of n.

Sol. The equation of curve is

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$$

or $\frac{x^n}{a^n} + \frac{y^n}{b^n} = 2$

Differentiating both sides w.r.t. x,

$$\frac{nx^{n-1}}{a^n} + \frac{ny^{n-1}}{b^n}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{b^n}{a^n}\frac{x^n - 1}{y^n - 1}$$

At (a, b), $\frac{dy}{dx} = -\frac{b^n}{a^n} \cdot \frac{a^{n-1}}{b^{n-1}} = -\frac{b}{a}$

$\therefore$ slope of tangent at $(a, b) = -\frac{b}{a}$

$\therefore$ equation of tangent at (a, b) is

$$y - b = -\frac{b}{a}(x - a)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

or $\quad \dfrac{y}{b} - 1 = -\dfrac{x}{a} + 1$

or $\quad \dfrac{x}{a} + \dfrac{y}{b} = 2$

Hence the result.

Example 6. Prove that the line $l\,x + m\,y + n = 0$ is a normal to $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

if $\dfrac{a^2}{l^2} + \dfrac{b^2}{m^2} = \dfrac{(a^2 - b^2)^2}{n^2}$.

Sol. The equation of curve is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ ...(1)

Let $l\,x + m\,y + n = 0$ be normal to curve at (x_1, y_1)

$\therefore \quad (x_1, y_1)$ lies on (1)

$\therefore \quad \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} = 1$ ...(2)

Differentiating both sides of (1) w.r.t. x,

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}$$

At (x_1, y_1), $\dfrac{dy}{dx} = -\dfrac{b^2}{a^2}\dfrac{x_1}{y_1}$, which is slope of tangent.

$\therefore$ at (x_1, y_1), slope of normal $= \dfrac{a^2}{b^2}\dfrac{y_1}{x_1}$

$\therefore$ equation of normal at (x_1, y_1) is

$$y - y_1 = \frac{a^2}{b^2}\frac{y_1}{x_1}(x - x_1)$$

or $\quad b^2 x_1 y - b^2 x_1 y_1 = a^2 y_1 x - a^2 x_1 y_1$

or $\quad a^2 y_1 x - b^2 x_1 y - (a^2 - b^2) x_1 y_1 = 0$ ...(3)

The given equation of normal is

$l\,x + m\,y + n = 0$ ...(4)

Now (3) and (4) represent the same line.

$$\therefore \quad \frac{a^2 y_1}{l} = \frac{-b^2 x_1}{m} = \frac{-(a^2 - b^2) x_1 y_1}{n}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\therefore \quad \frac{a^2}{l} = \frac{-(a^2-b^2)x_1}{n}, \quad \frac{b^2}{m} = \frac{(a^2-b^2)y_1}{n}$$

$$\Rightarrow \quad x_1 = -\frac{a^2 n}{l(a^2-b^2)}, \quad y_1 = \frac{b^2 n}{m(a^2-b^2)}$$

Putting values of x_1, y_1 in (2), we get,

$$\frac{a^4 n^2}{a^2 \cdot l^2 (a^2-b^2)^2} + \frac{b^4 n^2}{b^2 \cdot m^2 (a^2-b^2)^2} = 1$$

$$\frac{a^2 n^2}{l^2 (a^2-b^2)^2} + \frac{b^2 n^2}{m^2 (a^2-b^2)^2} = 1$$

or $$\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2-b^2)^2}{n^2}$$

which is required condition.

Example 7. Show that the sum of the intercepts of the tangent at any point to the curve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ on the axes is constant.

Sol. The equation of given curve is

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}} \qquad \text{...(1)}$$

or $$\sqrt{x} + \sqrt{y} = \sqrt{a}$$

Let (x_1, y_1) be any point on it

$$\therefore \quad \sqrt{x_1} + \sqrt{y_1} = \sqrt{a} \qquad \text{...(2)}$$

Differentiating both sides of (1) w.r.t. x,

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

At (x_1, y_1), $\frac{dy}{dx} = -\frac{\sqrt{y_1}}{\sqrt{x_1}}$, which is slope of tangent.

$\therefore$ equation of tangent at (x_1, y_1) is

$$y - y_1 = -\frac{\sqrt{y_1}}{\sqrt{x_1}}(x - x_1)$$

or $$\frac{y}{\sqrt{y_1}} - \frac{y_1}{\sqrt{y_1}} = -\frac{x}{\sqrt{x_1}} + \frac{x_1}{\sqrt{x_1}}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

or $\dfrac{x}{\sqrt{x_1}} + \dfrac{y}{\sqrt{y_1}} = \sqrt{x_1} + \sqrt{y_1}$

or $\dfrac{x}{\sqrt{x_1}} + \dfrac{y}{\sqrt{y_1}} = \sqrt{a}$ [$\because$ of (2)]

or $\dfrac{x}{\sqrt{a}\sqrt{x_1}} + \dfrac{y}{\sqrt{a}\sqrt{y_1}} = 1$

which is in intercept form

$\therefore$ intercept on x-axis $= \sqrt{a}\sqrt{x_1}$

and intercept on y-axis $= \sqrt{a}\sqrt{y_1}$

$\therefore$ sum of intercepts on axes $= \sqrt{a}\sqrt{x_1} + \sqrt{a}\sqrt{y_1}$

$= \sqrt{a}(\sqrt{x_1} + \sqrt{y_1})$

$= \sqrt{a}(\sqrt{a})$ [$\because$ of (2)]

$= a$, which is constant

Hence the result.

Example 8. Show that the line $x\cos\alpha + y\sin\alpha = p$ is a tangent to the curve

$$\left(\frac{x}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y}{b}\right)^{\frac{n}{n-1}} = 1$$

if $(a\cos\alpha)^n + (b\sin\alpha)^n = p^n$

Sol. The equation of curve is

$$\left(\frac{x}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y}{b}\right)^{\frac{n}{n-1}} = 1 \quad ...(1)$$

The equation of line is

$$x\cos\alpha + y\sin\alpha - p = 0 \quad ...(2)$$

Let line (2) touch (1) at (x_1, y_1)

Since (x_1, y_1) lies on (1)

$$\therefore \left(\frac{x_1}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y_1}{b}\right)^{\frac{n}{n-1}} = 1 \quad ...(3)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Differentiating both sides of (1) w.r.t. x,

$$\frac{n}{n-1}\left(\frac{x}{a}\right)^{\frac{1}{n-1}}\cdot\frac{1}{a}+\frac{n}{n-1}\left(\frac{y}{b}\right)^{\frac{1}{n-1}}\cdot\frac{1}{b}\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{b}{a}\left(\frac{bx}{ay}\right)^{\frac{1}{n-1}}$$

At (x_1, y_1), $\frac{dy}{dx}=-\frac{b}{a}\left(\frac{bx_1}{ay_1}\right)^{\frac{1}{n-1}}$, which is slope of tangent.

$\therefore$ equation of tangent at (x_1, y_1) is

$$y-y_1=-\frac{b}{a}\left(\frac{bx_1}{ay_1}\right)^{\frac{1}{n-1}}(x-x_1)$$

or $$\frac{(y-y_1)y_1^{\frac{1}{n-1}}}{b^n}=-\frac{x_1^{\frac{1}{n-1}}}{a^n}(x-x_1)$$

or $$\frac{1}{b}\left(\frac{y_1}{b}\right)^{\frac{1}{n-1}}y-\left(\frac{y_1}{b}\right)^{\frac{n}{n-1}}=-\frac{1}{a}\left(\frac{x_1}{a}\right)^{\frac{1}{n-1}}x+\left(\frac{x_1}{a}\right)^{\frac{n}{n-1}}$$

or $$\frac{1}{a}\left(\frac{x_1}{a}\right)^{\frac{1}{n-1}}x+\frac{1}{b}\left(\frac{y_1}{b}\right)^{\frac{1}{n-1}}y=\left(\frac{x_1}{a}\right)^{\frac{n}{n-1}}+\left(\frac{y_1}{b}\right)^{\frac{n}{n-1}}$$

or $$\frac{1}{a}\left(\frac{x_1}{a}\right)^{\frac{1}{n-1}}x+\frac{1}{b}\left(\frac{y_1}{b}\right)^{\frac{1}{n-1}}y=1 \qquad [\because \text{ of } (3)]$$

or $$\frac{1}{a}\left(\frac{x_1}{a}\right)^{\frac{1}{n-1}}x+\frac{1}{b}\left(\frac{y_1}{b}\right)^{\frac{1}{n-1}}y-1=0 \qquad ...(4)$$

Nor (4) and (2) represent the same line

$$\therefore \quad \frac{\frac{1}{a}\left(\frac{x_1}{a}\right)^{\frac{1}{n-1}}}{\cos\alpha}=\frac{\frac{1}{b}\left(\frac{y_1}{b}\right)^{\frac{1}{n-1}}}{\sin\alpha}=\frac{-1}{-p}$$

$$\therefore \quad \left(\frac{x_1}{a}\right)^{\frac{1}{n-1}}=\frac{a\cos\alpha}{p},\ \left(\frac{y_1}{b}\right)^{\frac{1}{n-1}}=\frac{b\sin\alpha}{p}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\Rightarrow \quad \left[\left(\frac{x_1}{a}\right)^{\frac{1}{n-1}}\right]^n = \left(\frac{a\cos\alpha}{p}\right)^n, \left[\left(\frac{y_1}{b}\right)^{\frac{1}{n-1}}\right]^n = \left(\frac{b\sin\alpha}{p}\right)^n$$

$$\Rightarrow \quad \left(\frac{x_1}{a}\right)^{\frac{n}{n-1}} = \frac{(a\cos\alpha)^n}{p^n}, \left(\frac{y_1}{b}\right)^{\frac{n}{n-1}} = \frac{(b\sin\alpha)^n}{p^n}$$

$$\Rightarrow \quad \left(\frac{x_1}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y_1}{b}\right)^{\frac{n}{n-1}} = \frac{(a\cos\alpha)^n}{p^n} + \frac{(b\sin\alpha)^n}{p^n}$$

$$\Rightarrow \quad 1 = \frac{(a\cos\alpha)^n}{p^n} + \frac{(b\sin\alpha)^n}{p^n} \qquad [\because \text{ of } (3)]$$

$$\Rightarrow \quad (a\cos\alpha)^n + (b\sin\alpha)^n = p^n$$

which is required condition.

EXERCISE 6 (*b*)

1. Find the equation of the tangent and normal to the curve $y^2 = \dfrac{x^2}{3-x}$ at $(2, -2)$.
2. Find the equations of normal lines to the curve $y = x^3 - 3x$ which are parallel to the line $x + 9y = 14$.
3. Find the equation of the tangent and normal to the parabola $y^2 = 4ax$ at (x_1, y_1).
4. Find the equation of tangent and normal to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at (x_1, y_1).
5. Find the equation of tangent and normal to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at (x_0, y_0).
6. Find the equations of the tangent and normal to the curve $x = at^2$, $y = 2at$ at the point 't'.
7. Find the equation of the tangent and normal to the parabola $y^2 = 4ax$ at $(am^2, -2am)$.
8. Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.
9. Find the equation of the tangent and normal to the curve $x = a\cos\theta$, $y = b\sin\theta$ at the point θ.
10. Find the equation of the tangent and normal to the curve $x = \cos t$, $y = \sin t$ at $t = \dfrac{\pi}{4}$.
11. Find the equation of the normal at point 'θ' on the curve $x = 3\cos\theta - \cos^3\theta$, $y = 3\sin\theta - \sin^3\theta$ and show that at the point where $\theta = \dfrac{\pi}{4}$, the normal passes through the origin.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

12. Find the equation of the tangent to the curve

$$y(x-2)(x-3)-x+7=0$$

at the point, where it cuts the x-axis.

13. If the normal to the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=\frac{2}{3}$ makes an angle θ with the axis of x, show that its equation is $y\cos\theta - x\sin\theta = a\cos 2\theta$.

14. Prove that the line $y = x + a\sqrt{2}$ touches the circle $x^2 + y^2 = a^2$ at the point $\left(-\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$.

15. Prove that all points of the curve

$$y^2 = 4a\left(x + a\sin\frac{x}{a}\right)$$

at which the tangent is parallel to the axis of x lie on a parabola.

16. Tangents are drawn from the origin to the curve $y = \sin x$. Prove that their points of contact lie on the curve $x^2y^2 = x^2 - y^2$.

17. Find the equation of the tangent to the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ and show that the portion of the tangent intercepted between the co-ordinate axes is constant.

18. Prove that the portion of the tangent intercepted between the point of contact and the x-axis in the curve.

$$x = a\left(\cos t + \log\tan\frac{t}{2}\right),\ y = a\sin t \text{ is constant.}$$

19. Prove that the portion of the tangent of the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ at the point θ, intercepted between the axes is of constant length.

20. Prove that in the curve $y = \frac{c}{2}\left[e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right]$, the length of the perpendicular, from the foot of the ordinate of a point P on the curve, upon the tangent at P is constant.

21. Find the length of perpendicular from the origin to the tangent at any point on the curve $x^m y^n = a^{m+n}$.

22. Find the condition that the line $lx + my + n = 0$ may touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

23. Find the condition that the straight line $x\cos\alpha + y\sin\alpha = p$ may be a tangent to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

24. If $l\,x + m\,y = 1$ touches the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 1$, show that $(al)^{\frac{n}{n-1}} + (bm)^{\frac{n}{n-1}} = 1$.

25. Show that the line $x\cos\theta + y\sin\theta - p = 0$ will touch the curve $x^m y^n = a^{m+n}$ if $p^{m+n} \cdot m^m \cdot n^n = (m+n)^{m+n} \cdot a^{m+n} \cos^m\theta \sin^n\theta$

ANSWERS

1. $2x + y - 2 = 0$; $x - 2y - 6 = 0$

2. $x + 9y - 20 = 0, x + 9y + 20 = 0$

3. $y y_1 = 2a(x + x_1)$; $y - y_1 = -\frac{y_1}{2a}(x - x_1)$

4. $\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1$; $\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$

5. $\frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1$; $\frac{a^2 x}{x_0} + \frac{b^2 y}{y_0} = a^2 + b^2$

6. $x - ty + a t^2 = 0$; $tx + y - 2at - at^3 = 0$

7. $x + my + a m^2 = 0$; $y = mx - 2am - a m^3$

8. $2x + 3my - a m^2 (3m^2 + 2) = 0$

9. $\frac{x\cos\theta}{a^2} + \frac{y\sin\theta}{b^2} = 1$; $ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2 - b^2$

10. $x + y - \sqrt{2} = 0$; $x - y = 0$

12. $x - 20y - 7 = 0$

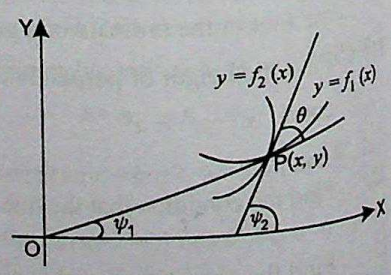
21. $\frac{(m+n)xy}{\sqrt{m^2 y^2 + n^2 x^2}}$

22. $a^2 l^2 + b^2 m^2 = n^2$

23. $a^2\cos^2\alpha - b^2\sin^2\alpha = p^2$

Art-4. Angle of Intersection of Two Curves

Def. The angle of intersection of two curves at a point of intersection is the angle between the tangents to the two curves at that point.



Let θ be the angle of intersection of two curves $y = f_1(x)$ and $y = f_2(x)$ at the point P (x, y). Let m_1, m_2 be the slopes of the tangents to the two curves at P.

$$\therefore \quad \tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Condition of Tangency

If two curves touch each other, then $\theta = 0$

$$\therefore \quad \tan\theta = 0 \quad \Rightarrow \quad \frac{m_1 - m_2}{1 + m_1 m_2} = 0$$

$$\Rightarrow \quad m_1 - m_2 = 0 \quad \Rightarrow \quad m_1 = m_2$$

$\therefore$ *when two curves touch each other, the slopes of the tangents to the two curves at a point are equal.*

Condition of Orthogonality

Two curves are said to cut orthogonally if angle of intersection between them is $\frac{\pi}{2}$.

$\therefore$ in case of orthogonality,

$$\theta = \frac{\pi}{2} \quad \Rightarrow \quad \cot\theta = 0 \quad \Rightarrow \quad \frac{1 + m_1 m_2}{m_1 - m_2} = 0$$

$$\Rightarrow \quad 1 + m_1 m_2 = 0 \quad \Rightarrow \quad m_1 m_2 = -1$$

$\therefore$ *if two curves cut orthogonally, product of the slopes of the tangents to the curves at the point of intersection = –1.*

Note : Working Rule to find the angle of intersection.

(*i*) Find the point of intersection of two curves by solving their equations simultaneously.

(*ii*) Find slopes m_1, m_2 of two curves at the point of intersection where $m = \frac{dy}{dx}$

(*iii*) Apply $\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$ to find angle θ.

ILLUSTRATIVE EXAMPLES

Example 1. Find the angle of intersection of the curves

$$y^2 = 2x \text{ and } x^2 + y^2 = 8$$

Sol. The equations of the curves are

$$y^2 = 2x \quad \text{...(1)}$$

and $$x^2 + y^2 = 8 \quad \text{...(2)}$$

From (1) and (2), we get,

$$x^2 + 2x = 8 \quad \text{or} \quad x^2 + 2x - 8 = 0$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$\therefore \quad (x-2)(x+4)=0 \quad \Rightarrow \quad x=2, -4$

When $x=2$, from (2), $y^2=4 \quad \Rightarrow \quad y=-2, 2$

$\therefore$ points of intersection are $(2, -2), (2, 2)$

When $x=-4$, from (1), $y^2=-8 \quad \Rightarrow$ y is imaginary.

$\therefore$ there is no point of intersection in this case.

$\therefore$ $(2, 2), (2, -2)$ are only two points of intersection.

Take the point (2, 2)

From (1), $2y\dfrac{dy}{dx}=2 \quad \Rightarrow \quad \dfrac{dy}{dx}=\dfrac{1}{y}$

At (2, 2), $\dfrac{dy}{dx}=\dfrac{1}{2} \quad \Rightarrow \quad m_1=\dfrac{1}{2}$

From (2), $2x+2y\dfrac{dy}{dx}=0 \quad \Rightarrow \quad \dfrac{dy}{dx}=-\dfrac{x}{y}$

At (2, 2), $\dfrac{dy}{dx}=-\dfrac{2}{2}=-1 \quad \Rightarrow \quad m_2=-1$

Now $\tan\theta_1=\dfrac{m_1-m_2}{1+m_1m_2}$

$$=\frac{\frac{1}{2}+1}{1+\left(\frac{1}{2}\right)(-1)}=\frac{\frac{3}{2}}{\frac{1}{2}}=3$$

$\therefore \quad \theta_1=\tan^{-1}(3)$

Take the point (2, –2)

From (1), $\dfrac{dy}{dx}=\dfrac{1}{y}$

At (2, –2), $\dfrac{dy}{dx}=-\dfrac{1}{2} \quad \Rightarrow \quad m_1=-\dfrac{1}{2}$

From (2), $\dfrac{dy}{dx}=-\dfrac{x}{y}$

At (2, –2), $\dfrac{dy}{dx}=-\dfrac{2}{-2}=1 \quad \Rightarrow \quad m_2=1$

Now $\tan\theta_2=\left|\dfrac{m_1-m_2}{1+m_1m_2}\right|=\left|\dfrac{-\frac{1}{2}-1}{1+\left(-\frac{1}{2}\right)(1)}\right|=\left|\dfrac{-\frac{3}{2}}{\frac{1}{2}}\right|=3$

$\therefore \quad \theta_2=\tan^{-1}(3)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 2. Show that the circles

$$x^2+y^2+2ax+c=0 \quad \text{and} \quad x^2+y^2+2by+c=0$$

touch if $\dfrac{1}{a^2}+\dfrac{1}{b^2}=\dfrac{1}{c}$

Sol. The equations of two circles are

$$x^2+y^2+2ax+c=0 \qquad ...(1)$$

and $$x^2+y^2+2by+c=0 \qquad ...(2)$$

Let (x_1, y_1) be the point of intersection of circles

$$\therefore \quad x_1^2+y_1^2+2ax_1+c=0 \qquad ...(3)$$

and $$x_1^2+y_1^2+2by_1+c=0 \qquad ...(4)$$

From (1), $2x+2y\dfrac{dy}{dx}+2a=0 \quad \Rightarrow \quad \dfrac{dy}{dx}=-\dfrac{x+a}{y}$

At (x_1, y_1) $\dfrac{dy}{dx}=-\dfrac{x_1+a}{y_1} \quad \Rightarrow \quad m_1=-\dfrac{x_1+a}{y_1}$

From (2), $2x+2y\dfrac{dy}{dx}+2b\dfrac{dy}{dx}=0 \quad \Rightarrow \quad \dfrac{dy}{dx}=-\dfrac{x}{y+b}$

At (x_1, y_1), $\dfrac{dy}{dx}=-\dfrac{x_1}{y_1+b} \quad \Rightarrow \quad m_2=-\dfrac{x_1}{y_1+b}$

Since two circles touch

$$\therefore \quad m_1=m_2 \quad \Rightarrow \quad -\frac{x_1+a}{y_1}=-\frac{x_1}{y_1+b}$$

$$\Rightarrow \quad (x_1+a)(y_1+b)=x_1y_1 \quad \Rightarrow \quad bx_1+ay_1+ab=0 \qquad ...(5)$$

Subtracting (4) from (3), we get,

$$2ax_1-2by_1=0 \quad \Rightarrow \quad ax_1=by_1 \quad \Rightarrow \quad x_1=\frac{by_1}{a} \qquad ...(6)$$

From (5) and (6), we get,

$$\frac{b^2y_1}{a}+ay_1+ab=0 \quad \Rightarrow \quad (a^2+b^2)y_1=-a^2b \quad \Rightarrow \quad y_1=-\frac{a^2b}{a^2+b^2}$$

$\therefore$ from (6), $x_1=\dfrac{b}{a}\left(-\dfrac{a^2b}{a^2+b^2}\right)=-\dfrac{ab^2}{a^2+b^2}$

$\therefore$ point of contact of two circles is $\left(-\dfrac{ab^2}{a^2+b^2}, -\dfrac{a^2b}{a^2+b^2}\right)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Putting values of x_1, y_1 in (3), we get,

$$\frac{a^2 b^4}{(a^2+b^2)^2} + \frac{a^4 b^2}{(a^2+b^2)^2} + 2a \cdot \left(-\frac{ab^2}{a^2+b^2}\right) + c = 0$$

or $$\frac{a^2 b^4 + a^4 b^2}{(a^2+b^2)^2} - \frac{2a^2 b^2}{a^2+b^2} + c = 0$$

or $$\frac{a^2 b^2}{a^2+b^2} - \frac{2a^2 b^2}{a^2+b^2} + c = 0$$

or $$c = \frac{a^2 b^2}{a^2+b^2} \Rightarrow \frac{1}{c} = \frac{a^2+b^2}{a^2 b^2}$$

or $$\frac{1}{c} = \frac{1}{a^2} + \frac{1}{b^2}$$

which is the required condition.

Example 3. Show that the curves $l x^2 + m y^2 = 1$ and $l' x^2 + m' y^2 = 1$ will intersect orthogonally if $\frac{1}{l} - \frac{1}{m} = \frac{1}{l'} - \frac{1}{m'}$.

Sol. The equations of curves are

$$l x_1^2 + m y^2 = 1 \quad ...(1)$$

and $$l' x_1^2 + m' y^2 = 1 \quad ...(2)$$

Let (x_1, y_1) be the point of intersection of two curves.

$$\therefore \quad l x_1^2 + m y_1^2 = 1 \quad ...(3)$$

and $$l' x_1^2 + m' y_1^2 = 1 \quad ...(4)$$

Subtracting (4) from (3), we get,

$$(l - l') x_1^2 + (m - m') y_1^2 = 0$$

$$\Rightarrow (l - l') x_1^2 = -(m - m') y_1^2 \quad ...(5)$$

From (1), $2 l x + 2 m y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{l x}{m y}$

At (x_1, y_1), $\frac{dy}{dx} = -\frac{l x_1}{m y_1} \Rightarrow m_1 = -\frac{l x_1}{m y_1}$

From (2), $2 l' x + 2 m' y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{l' x}{m' y}$

At (x_1, y_1), $\frac{dy}{dx} = -\frac{l' x_1}{m' y_1} \Rightarrow m_2 = -\frac{l' x_1}{m' y_1}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Since two curves cut orthogonally

$\therefore\ m_1 m_2 = -1$

$$\therefore\ \left(-\frac{l x_1}{m y_1}\right)\left(-\frac{l' x_1}{m' y_1}\right) = -1$$

$$\Rightarrow\ l l' x_1^2 = -m m' y_1^2 \qquad ...(6)$$

From (5) and (6), we get,

$$\frac{l-l'}{l l'} = \frac{m-m'}{m m'} \quad \text{or} \quad \frac{1}{l'} - \frac{1}{l} = \frac{1}{m'} - \frac{1}{m}$$

$$\text{or} \quad \frac{1}{l} - \frac{1}{m} = \frac{1}{l'} - \frac{1}{m'}$$

which is required condition.

EXERCISE 6 (*c*)

1. Find the angle of intersection between the curves

   $y = 4 - x^2$ and $y = x^2$

2. Find the angle of intersection between the curves

   (*i*) $x^2 + y^2 = 2a^2$ and $y^2 = ax$

   (*ii*) $y = x^3$ and $6y = 7 - x^2$

   (*iii*) $x^2 + y^2 = 2a^2$ and $xy = a^2$

   (*iv*) $x^2 + y^2 = 8$ and $xy = 4$

   (*v*) $y^2 = 4ax$ and $x^2 = 4by$

   (*vi*) $y^2 = x$ and $x^2 = y$

3. Find the angle of intersection of the curves $x^2 - y^2 = a^2$ and $x^2 + y^2 = a^2\sqrt{2}$.

4. Show that the circles $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 6x - 6y + 2 = 0$ touch each other. Find the coordinates of the point of contact and the equation of the tangent there.

5. Show that the curves $x^2 + 4y^2 = 8$ and $x^2 - 2y^2 = 4$ cut each other orthogonally.

6. Show that the curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 = 2$ cut orthogonally.

7. Find the condition that the curves $ax^2 + by^2 = 1$ and $a_1x^2 + b_1y^2 = 1$ should cut orthogonally.

8. Find the condition that the curves $\frac{x^2}{a} + \frac{y^2}{b} = 1$ and $\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1$ may cut orthogonally.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

ANSWERS

1. $\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$ at $(-\sqrt{2}, 2)$; $\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$ at $(\sqrt{2}, 2)$

2. (*i*) $\tan^{-1} 3$ (*ii*) $90°$ (*iii*) Touch each other

(*iv*) Touch each other (*v*) $90°$, $\tan^{-1}\left[\frac{3a^{\frac{1}{3}}b^{\frac{1}{3}}}{2\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)}\right]$

(*vi*) $90°$, $\tan^{-1}\frac{3}{4}$

3. $45°$

4. $\left(\frac{1}{5}, \frac{3}{5}\right)$, $4x - 3y + 1 = 0$

7. $\frac{1}{a} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{b_1}$

8. $a - b = \alpha - \beta$

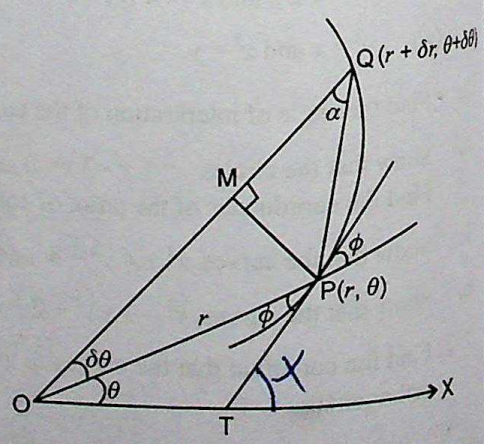
Art-5. Angle between Radius Vector and Tangent

Find the angle between the radius vector and the tangent at any point (r, θ) of the given curve.

OR

With usual notations, prove that $\tan\phi = r\frac{d\theta}{dr}$. (J.U. 2016)

Proof : Let P(r, θ) be any point on the given curve and Q $(r + \delta r, \theta + \delta\theta)$ be any other point on the curve in the neighbourhood of P.



Let the tangent to the curve at P meet OX in T such that $\angle OPT = \phi$.

From P, draw $\perp$ PM OQ.

Let $\angle PQM = \alpha$

In rt. $\angle d$ ΔOMP,

$$\frac{OM}{OP} = \cos\delta\theta$$

or $$\frac{OM}{r} = \cos\delta\theta$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$\therefore \quad OM = r\cos\delta\theta$ ...(1)

and $\dfrac{PM}{OP} = \sin\delta\theta$ or $\dfrac{PM}{r} = \sin\delta\theta$

$\therefore \quad PM = r\sin\delta\theta$ ...(2)

$MQ = OQ - OM = r + \delta r - r\cos\delta\theta$ [$\because$ of (1)]

$= r(1-\cos\delta\theta) + \delta r = 2r\sin^2\dfrac{\delta\theta}{2} + \delta r$ ...(3)

$\therefore$ in rt. $\angle d\ \Delta PMQ$,

$$\tan\alpha = \frac{PM}{MQ} = \frac{r\sin\delta\theta}{2r\sin^2\dfrac{\delta\theta}{2} + \delta r}$$ [$\because$ of (2), (3)]

$$\therefore \quad \tan\alpha = \frac{r\dfrac{\sin\delta\theta}{\delta\theta}}{\dfrac{r\sin\dfrac{\delta\theta}{2}}{\dfrac{\delta\theta}{2}}.\sin\dfrac{\delta\theta}{2} + \dfrac{\delta r}{\delta\theta}}$$

Let $Q \to P$. Therefore $\alpha \to \phi, \delta\theta \to 0$

[$\because$ PQ becomes tangent PT and OQ coincides with OP]

$$\therefore \quad \tan\theta = \underset{\delta\theta\to 0}{\text{Lt}} \frac{r\dfrac{\sin\delta\theta}{\delta\theta}}{r\dfrac{\sin\dfrac{\delta\theta}{2}}{\dfrac{\delta\theta}{2}}.\sin\dfrac{\delta\theta}{2} + \dfrac{\delta r}{\delta\theta}}$$

$$\Rightarrow \quad \tan\phi = \frac{r.1}{r.1.0 + \dfrac{dr}{d\theta}} \qquad \left[\because \underset{\theta\to 0}{\text{Lt}} \frac{\sin\theta}{\theta} = 1\right]$$

$$\Rightarrow \quad \tan\phi = \frac{r}{\dfrac{dr}{d\theta}}$$

$$\Rightarrow \quad \tan\phi = r\frac{d\theta}{dr}$$

which gives angle ϕ between radius vector and the tangent at any point.

Note 1. ϕ is the angle between the radius vector and the tangent.

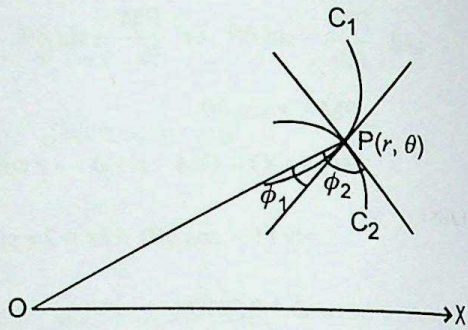
Note 2. $\psi = \theta + \phi$, where ψ is the angle which the tangent makes with OX.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Note 3. Angle of intersection of two curves

Let ϕ_1 and ϕ_2 be the angles which the tangents to the two curves C_1 and C_2 make with radius vector OP.



$\therefore$ angle of intersection of two curves = angle between the tangents at P

$= \phi_2 - \phi_1$.

Note 4. For orthogonal intersection

$$\alpha = \frac{\pi}{2} \Rightarrow \phi_1 = \frac{\pi}{2} + \phi_2$$

$$\therefore \quad \tan\phi_1 = \tan\left(\frac{\pi}{2} + \phi_2\right) = -\cot\phi_2$$

$$\therefore \quad \tan\phi_1 = -\frac{1}{\tan\phi_2} \qquad \Rightarrow \quad \tan\phi_1 \tan\phi_2 = -1$$

$\therefore$ $\tan\phi$ and $\tan\phi_2$ for the two curves will be negative reciprocals of each other.

ILLUSTRATIVE EXAMPLES

Example 1. Show that in the parabola $\frac{2a}{r} = 1 - \cos\theta$, $\phi = \pi - \frac{1}{2}\theta$.

Or

Find the angle at which the radius vector cuts the curve $\frac{2a}{r} = 1 - \cos\theta$.

Sol. The equation of given curve is $\frac{2a}{r} = 1 - \cos\theta$

$$\therefore \quad \log\left(\frac{2a}{r}\right) = \log(1 - \cos\theta)$$

$$\therefore \quad \log 2a - \log r = \log(1 - \cos\theta)$$

Differentiating both sides w.r.t θ, we get,

$$0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1 - \cos\theta}.\sin\theta$$

$$\therefore \quad -\frac{1}{r}\frac{dr}{d\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

 Gandhi Memorial College Of Education Bantalab Jammu

$\Rightarrow \quad -\cot\phi = \cot\dfrac{\theta}{2}$ $\qquad \left[\because \tan\phi = r\dfrac{d\theta}{dr}\right]$

$\Rightarrow \quad -\tan\phi = \tan\dfrac{\theta}{2}$ $\qquad \Rightarrow \quad \tan(\pi-\phi) = \tan\dfrac{\theta}{2}$

$\Rightarrow \quad \pi-\phi = \dfrac{\theta}{2}$ $\qquad \Rightarrow \quad \phi = \pi - \dfrac{1}{2}\theta$

Example 2. Find the angle between the radius vector and the tangent in the following case :

$$r = a(1+\cos\theta) \quad \text{at} \quad \theta = \frac{\pi}{2}.$$

Sol. The equation of curve is $r = a(1+\cos\theta)$ ...(1)

$\therefore \quad \dfrac{dr}{d\theta} = -a\sin\theta \quad \Rightarrow \quad \dfrac{d\theta}{dr} = -\dfrac{1}{a\sin\theta}$ ...(2)

Now $\tan\phi = r\dfrac{d\theta}{dr} = a(1+\cos\theta).\dfrac{-1}{a\sin\theta}$ $\qquad [\because$ of (1), (2)$]$

$$= -\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$\therefore \quad \phi = \dfrac{\pi}{2}+\dfrac{\theta}{2}$

At $\theta = \dfrac{\pi}{2}, \phi = \dfrac{\pi}{2}+\dfrac{\pi}{4} = \dfrac{3\pi}{4}$

Example 3. Find the angle between the tangent and the radius vector in the case of the curve $r^n = a^n \sec(n\theta+\alpha)$.

Sol. The equation of curve is $r^n = a^n \sec(n\theta+\alpha)$

$\therefore \quad \log r^n = \log a^n + \log\sec(n\theta+\alpha)$

$\therefore \quad n\log r = n\log a + \log\sec(n\theta+\alpha)$

Differentiating w.r.t θ, we get,

$$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\sec(n\theta+\alpha)}.\sec(n\theta+\alpha)\tan(n\theta+\alpha).n$$

$\therefore \quad \dfrac{1}{r}\dfrac{dr}{d\theta} = \tan(n\theta+\alpha) \Rightarrow \cot\phi = \tan(n\theta+\alpha)$

$\therefore \quad \tan\phi = \cot(n\theta+\alpha) \quad \Rightarrow \tan\phi = \tan\left[\dfrac{\pi}{2}-(n\theta+\alpha)\right]$

$\Rightarrow \quad \phi = \dfrac{\pi}{2} - (n\theta+\alpha)$, which is required angle.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 4. Find the angle between the curves $r^m = a^m \sin m\theta$ and $r^m = a^m \cos m\theta$

Sol. The equations of curves are

$$r^m = a^m \sin m\theta \quad \ldots(1)$$

and $$r^m = a^m \cos m\theta$$

From (1), we get

$$\log r^m = \log (a^m \sin m\theta)$$

$$\therefore \quad \log r^m = \log a^m + \log \sin m\theta$$

$$\therefore \quad m \log r = m \log a + \log \sin m\theta$$

Differentiating w.r.t. θ,

$$\frac{m}{r}\frac{dr}{d\theta} = 0 + \frac{m \cos m\theta}{\sin m\theta}$$

$$\therefore \quad r\frac{d\theta}{dr} = \tan m\theta$$

$$\Rightarrow \quad \tan \phi_1 = \tan m\theta$$

From (2), we get

$$\log r^m = \log (a^m \cos m\theta)$$

$$\therefore \quad \log r^m = \log a^m + \log \cos m\theta$$

$$\therefore \quad m \log r = m \log a + \log \cos m\theta$$

Differentiating w.r.t. θ,

$$\frac{m}{r}\frac{dr}{d\theta} = 0 + \frac{-m \sin m\theta}{\cos m\theta}$$

$$\therefore \quad r\frac{d\theta}{dr} = -\cot m\theta$$

$$\therefore \quad \tan \phi_2 = -\cot m\theta$$

$$\therefore \quad \tan \phi_1 . \tan \phi_2 = (\tan m\theta)(-\cot m\theta) = -1$$

$\therefore$ curves cut orthogonally.

Example 5. Find the angle of intersection between the curves

$$r = \frac{a\theta}{1+\theta} \text{ and } r = \frac{a}{(1+\theta)^2}$$

Sol. The equations of curves are

$$r = \frac{a\theta}{1+\theta} \quad \ldots(1)$$

and $$r = \frac{a}{(1+\theta)^2} \quad \ldots(2)$$

From (1) and (2), we get

$$\frac{a\theta}{1+\theta} = \frac{a}{(1+\theta)^2} \quad \Rightarrow \quad \frac{\theta}{1} = \frac{1}{1+\theta}$$

$$\Rightarrow \quad \theta(\theta+1) = 1 \quad \ldots(3)$$

$$\Rightarrow \quad \theta^2 + \theta - 1 = 0$$

$$\therefore \quad \theta = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

 Gandhi Memorial College Of Education Bantalab Jammu

From (1), $\dfrac{dr}{d\theta} = a\left[\dfrac{(1+\theta).1-\theta.1}{(1+\theta)^2}\right] = \dfrac{a}{(1+\theta)^2}$

$\therefore \quad r\dfrac{d\theta}{dr} = \dfrac{a\,\theta}{1+\theta}.\dfrac{(1+\theta)^2}{a} = \theta\,(1+\theta)$

$\therefore \quad \tan\phi_1 = 1$ [$\because$ of (3)]

$\therefore \quad \phi_1 = \dfrac{\pi}{4}$

From (2), $\dfrac{dr}{d\theta} = -\dfrac{2\,a}{(1+\theta)^3}$

$\therefore \quad r\,\dfrac{d\theta}{dr} = \dfrac{a}{(1+\theta)^2}.\dfrac{(1+\theta)^3}{-2\,a}$

$\therefore \quad \tan\phi = -\dfrac{1+\theta}{2} = -\dfrac{1}{2\theta}$ [$\because$ of (3)]

At $\quad \theta = \dfrac{-1\pm\sqrt{5}}{2}$,

$\tan\phi_2 = \tan\alpha$ (say)

$\Rightarrow \quad \phi_2 = \alpha$

$\therefore$ required angle $= |\,\phi_1 - \phi_2\,| = \left|\dfrac{\pi}{4} - \alpha\right|$

Example 6. Prove that the curves $r^2 = a^2\cos 2\theta$ and $r = a\,(1+\cos\theta)$ intersect at an angle $3\sin^{-1}\left(\dfrac{3}{4}\right)^{\frac{1}{4}}$.

Sol. The equations of curves are

$$r^2 = a^2\cos 2\theta \qquad \ldots(1)$$

and $$r = a\,(1+\cos\theta) \qquad \ldots(2)$$

From (1) and (2), we get

$$a^2\,(1+\cos\theta)^2 = a^2\cos 2\theta$$

$\Rightarrow \quad 1+\cos^2\theta + 2\cos\theta = 2\cos^2\theta - 1$

$\therefore \quad \cos^2\theta - 2\cos\theta - 2 = 0$

$\Rightarrow \quad \left(1-2\sin^2\dfrac{\theta}{2}\right)^2 - 2\left(1-2\sin^2\dfrac{\theta}{2}\right) - 2 = 0$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\Rightarrow \quad 1+4\sin^4\frac{\theta}{2}-4\sin^2\frac{\theta}{2}-2+4\sin^2\frac{\theta}{2}-2=0$$

$$\Rightarrow \quad 4\sin^4\frac{\theta}{2}=3$$

$$\Rightarrow \quad \sin\frac{\theta}{2}=\left(\frac{3}{4}\right)^{\frac{1}{4}} \quad \Rightarrow \quad \theta=2\sin^{-1}\left(\frac{3}{4}\right)^{\frac{1}{4}}$$

From (1), we get

$$\log r^2=\log a^2+\log\cos 2\theta$$

$$\therefore \quad 2\log r=2\log a+\log\cos 2\theta$$

Differentiating w.r.t. θ, we get,

$$\frac{2}{r}\frac{dr}{d\theta}=0+\frac{1}{\cos 2\theta}(-2\sin 2\theta)$$

$$\therefore \quad r\frac{d\theta}{dr}=-\cot 2\theta$$

$$\therefore \quad \tan\phi_1=\tan\left(\frac{\pi}{2}+2\theta\right)$$

$$\therefore \quad \phi_1=\frac{\pi}{2}+2\theta$$

From (2),

$$\frac{dr}{d\theta}=-a\sin\theta$$

$$\therefore \quad r\frac{d\theta}{dr}=a(1+\cos\theta).\left(\frac{1}{-a\sin\theta}\right)$$

$$\therefore \quad \tan\phi_2=-\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$\therefore \quad \tan\phi_2=-\cot\frac{\theta}{2}=\tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$\therefore \quad \phi_2=\frac{\pi}{2}+\frac{\theta}{2}$$

Required angle $=|\phi_1-\phi_2|$

$$=\frac{\pi}{2}+2\theta-\frac{\pi}{2}-\frac{\theta}{2}=\frac{3\theta}{2}=3\sin^{-1}\left(\frac{3}{4}\right)^{\frac{1}{4}} \qquad [\because \text{ of } (3)]$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

EXERCISE 6 (d)

1. In the curve $r = \frac{a}{\theta}$, show that $\tan\phi = -\theta$.

2. Find the angle at which the radius vector cuts the curve $\frac{2a}{r} = 1 + \cos\theta$.

3. For the curve $r = a(1 - \cos\theta)$, prove that $\phi = \frac{\theta}{2}$.

4. Find the angle between the radius vector and the tangent in each of the following cases :

   (*i*) $r = a(1 + \sin\theta)$ at $\theta = \frac{\pi}{6}$ (*ii*) $r^2 = a^2 \cos 2\theta$ at $\theta = \frac{\pi}{6}$

   (*iii*) $r = a \operatorname{cosec}^2 \frac{\theta}{2}$ at $\theta = \frac{\pi}{2}$ (*iv*) $r^m = a^m (\cos m\theta - \sin m\theta)$ at $\theta = 0$

5. Show that in the equiangular spiral $r = a\, e^{\theta \cot \alpha}$, the tangent is inclined at a constant angle to the radius vector.

6. Show that the radius vector is inclined at a constant angle to the tangent at any point on the equiangular spiral $r = a\, e^{b\theta}$.

7. Show that at any point P(r, θ) of the curve $\theta = \frac{\sqrt{r^2 - a^2}}{a} - \cos^{-1}\left(\frac{a}{r}\right)$

   (*i*) $\cos\phi = \frac{a}{r}$ (*ii*) $\tan\phi = \frac{\sqrt{r^2 - a^2}}{a}$

8. Find the angle at which the radius vector cuts the curve $\frac{l}{r} = 1 + e\cos\theta$

9. Prove that the tangent at any point (r, θ) on the curve $r^2 = a^2 \sin 2\theta$ makes an angle 3θ with the initial line.

10. Show that the angle between the tangent at any point P and the line joining P to the origin is the same at all points of the curve $\log(x^2 + y^2) = k \tan^{-1}\frac{y}{x}$.

11. If ϕ is the angle between the tangent to a curve and the radius vector drawn from the origin of co-ordinates to the point of contact, prove that

$$\tan\phi = \frac{x\frac{dy}{dx} - y}{x + y\frac{dy}{dx}}.$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Digitized by Eduvanam Bantalab Jammu

12. Prove that the spirals $r^n = a^n \cos n\theta$ and $r^n = b^n \sin n\theta$ intersect orthogonally.

13. Find the angle of intersection of the curves

$$r = a(1 + \cos\theta),\ r = b(1 - \cos\theta).$$

14. Find the angle of intersection of the curves

$$r = 2\sin\theta \text{ and } r = 2\cos\theta$$

15. Find the angle of intersection of the circle $r = a\cos\theta$ and cardioid $r = a(1 - \cos\theta)$.

16. Find the angle between the curves

(*i*) $r = a\theta, r\theta = a$ (*ii*) $r = 2a\cos\theta, r = 2a\sin\theta$

(*iii*) $r = a(1 + \sin\theta), r = a(1 - \sin\theta)$ (*iv*) $r = \sin\theta + \cos\theta, r = 2\sin\theta$

(*v*) $r = \sin\theta + \cos\theta, r = 2\cos\theta$ (*vi*) $r = a e^{\theta}, r e^{\theta} = b$

17. Find the angle of intersection of the parabolas

$$r = \frac{a}{1+\cos\theta} \text{ and } r = \frac{b}{1-\cos\theta}$$

18. Find the angle of intersection of the curves $r = a\cos\theta$ and $2r = a$.

19. Find the angle of intersection of the curves $r\sin\theta = 2a$ and $r = a\sec^2\frac{\theta}{2}$.

20. Show that the curves $r^n = a^n \sec(n\theta + \alpha)$ and $r^n = b^n \sec(n\theta + \beta)$ intersect at an angle which is independent of a and b.

21. Show that the circle $r = b$ cuts the curve

$$r^2 = a^2\cos 2\theta + b^2 = \text{at angle } \tan^{-1}\left(\frac{a^2}{b^2}\right).$$

ANSWERS

2. $\frac{\pi}{2} - \frac{\theta}{2}$

4. (*i*) $\frac{\pi}{3}$ (*ii*) $\frac{5\pi}{6}$ (*iii*) $\frac{3\pi}{4}$ (*iv*) $\frac{3\pi}{4}$

8. $\tan^{-1}\left(\frac{1+e\cos\theta}{e\sin\theta}\right)$ **13.** $\frac{\pi}{2}$ **14.** $\frac{\pi}{2}$ **15.** $\frac{\pi}{2} + \frac{\theta}{2}$

16. (*i*) $\frac{\pi}{2}$ (*ii*) $\frac{\pi}{2}$ (*iii*) $\frac{\pi}{2}$ (*iv*) $\frac{\pi}{4}$ (*v*) $\frac{\pi}{4}$ (*vi*) $\frac{\pi}{2}$

17. $\frac{\pi}{2}$ **18.** $\frac{\pi}{3}$ **19.** $\frac{\pi}{4}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

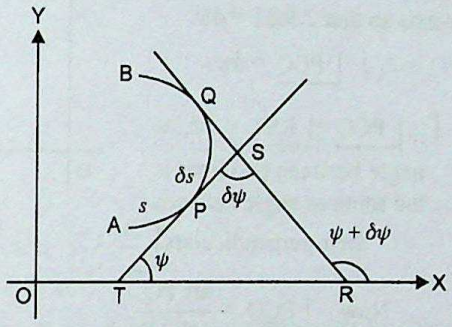
Gandhi Memorial College Of Education Bantalab Jammu

7

CURVATURE

Art-1. Definition of Curvature

Let P and Q be any two neighbouring points on a curve AB such that arc AP = s and arc AQ = $s + \delta s$ so that arc PQ = δs. Let the tangents to the curve at P and Q make angles ψ and $\psi + \delta\psi$ with x-axis so that $\angle RST = \delta\psi$. Then



(*i*) $\delta\psi$, measured in radians, is called the **total curvature** or **total bending** of the arc PQ,

(*ii*) the ratio $\dfrac{\delta\psi}{\delta s}$ is called the **average curvature** of the arc PQ,

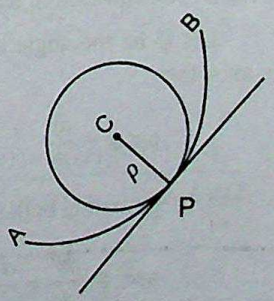
(*iii*) $\underset{\delta s \to 0}{\text{Lt}} \dfrac{\delta\psi}{\delta s}$, if it exists, is called the **curvature** of the curve at P and is denoted by κ

(*iv*) The reciprocal of curvature at any point P is called the **radius of curvature** and is denoted by Greek letter ρ

$$\rho = \frac{1}{\dfrac{d\psi}{ds}} = \frac{ds}{d\psi}$$

Art-2. Circle, Centre and Chord of Curvature

The centre of curvature of a curve at a point P is the point C which lies on the positive direction of the normal at P and which is at a distance ρ from it.



The circle with centre C and radius CP = ρ is called **circle of curvature** of the curve at P.

Any chord of the circle of curvature at P passing through P is called **chord of curvature** through P.

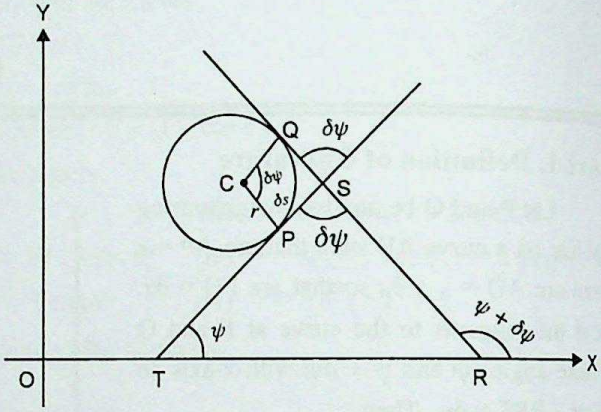
CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bhadrawah Jammu

Art-3. Curvature of a Circle

Prove that the curvature of a circle is constant and is equal to the reciprocal of the radius.

Proof. Consider a circle with centre C and radius = r. Let P and Q be any two points on it such that arc PQ = δs. Let ψ and $\psi + \delta\psi$ be the angles which the tangents PT and QR at P and Q make with x-axis so that $\angle RST = \delta\psi$.



$$\therefore \quad \angle PCQ = \delta\psi$$

$[\because \angle PCQ = \angle RST = \delta\psi$, as angle between two lines is the same as angle between their perpendiculars$]$

$$\text{Now} \quad \angle PCQ = \frac{\text{arc PQ}}{\text{CP}} \qquad \left[\because \theta = \frac{1}{r}\right]$$

$$\therefore \quad \delta\psi = \frac{\delta s}{r}$$

$$\Rightarrow \quad \frac{\delta\psi}{\delta s} = \frac{1}{r}$$

$$\therefore \quad \underset{\delta s \to 0}{\text{Lt}} \frac{\delta\psi}{\delta s} = \frac{1}{r} \quad \Rightarrow \quad \frac{d\psi}{ds} = \frac{1}{r}$$

$\Rightarrow$ curvature at any point P of a circle is the reciprocal of its radius and so it is constant.

Note. Radius of Curvature = Radius of Circle

Art-4. Radius of Curvature for Cartesian Curve $y = f(x)$

Find the radius of curvature at any point of the curve $y = f(x)$.

Proof. The equation of the curve is $y = f(x)$

Let ψ be the angle which the tangent to the curve at any point (x, y) makes with x-axis, then

$$\tan\psi = \frac{dy}{dx} \qquad \ldots(1)$$

Differentiating both sides w.r.t. s, we get,

$$\sec^2\psi \frac{d\psi}{ds} = \frac{d^2y}{dx^2}\frac{dx}{ds}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

or $\sec^2\psi \dfrac{d\psi}{ds} = \dfrac{d^2y}{dx^2}\cos\psi$ $\left[\because \dfrac{dx}{ds} = \cos\psi\right]$

$$\therefore \quad \frac{ds}{d\psi} = \frac{\sec^3\psi}{\dfrac{d^2y}{dx^2}} \quad \Rightarrow \quad \frac{ds}{d\psi} = \frac{(1+\tan^2\psi)^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$$

$$\Rightarrow \quad \rho = \frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}} \qquad [\because \text{ of } (1)$$

$$\Rightarrow \quad \rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} \quad \text{where } y_1 = \frac{dy}{dx} \text{ and } y_2 = \frac{d^2y}{dx^2}$$

Note : We know that the curvature of the curve at any point depends only upon its shape and so it is independent of the coordinate system. Therefore, interchanging x and y, we get,

$$\rho = \frac{(1+x_1^2)^{\frac{3}{2}}}{x_2} \quad \text{where } x_1 = \frac{dx}{dy} \quad \text{and} \quad x_2 = \frac{d^2x}{dy^2}$$

Art-5. Parametric Equations

Find the radius of curvature at any point of the curve $x = f(t), y = g(t)$.

Proof. The equations of curve are

$$x = f(t), \; y = g(t)$$

$$\therefore \quad \frac{dx}{dt} = f'(t), \; \frac{dy}{dt} = g'(t)$$

$$\text{Now} \quad \frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^2}\frac{dt}{dx} = \frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^2}\frac{1}{f'(t)}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Now $$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$\therefore \quad \rho = \frac{\left[1+\left\{\frac{g'(t)}{f'(t)}\right\}^2\right]^{\frac{3}{2}}.[f'(t)]^3}{f'(t)\,g''(t)-g'(t)\,f''(t)}$$

$$\therefore \quad \rho = \frac{[\{f'(t)\}^2+\{g'(t)\}^2]^{\frac{3}{2}}}{f'(t)\,g''(t)-g'(t)\,f''(t)}$$

Art-6. Find the formula for finding the radius of curvature when x and y are functions of s.

Proof : Let the equations of the curve be $x = f(s)\,,\, y = g(s)$

We know that $\frac{dx}{ds} = \cos\psi$ and $\frac{dy}{ds} = \sin\psi$

Now $\frac{d^2x}{ds^2} = -\sin\psi\frac{d\psi}{ds} = -\frac{\sin\psi}{\rho}$ and $\frac{d^2y}{ds^2} = \cos\psi\,\frac{d\psi}{ds} = \frac{\cos\psi}{\rho}$

$$\therefore \quad \left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2 = \frac{1}{\rho^2}(\sin^2\psi+\cos^2\psi) \quad \Rightarrow \quad \frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2$$

Cor. $$\therefore \quad \rho = -\frac{\sin\psi}{\frac{d^2x}{ds^2}} = -\frac{\frac{dy}{ds}}{\frac{d^2x}{ds^2}}$$

Also $$\rho = \frac{\cos\psi}{\frac{d^2y}{ds^2}} = \frac{\frac{dx}{ds}}{\frac{d^2y}{ds^2}}$$

Art-7. Radius of Curvature for Implicit Equation $f(x, y) = 0$

For implicit equation $f(x, y) = 0$, we have

$$\rho = \frac{[(f_x)^2+(f_y)^2]^{\frac{3}{2}}}{f_{xx}(f_y)^2-2f_{xy}f_xf_y+f_{yy}(f_x)^2}$$

where $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$, $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $f_{yy} = \frac{\partial^2 f}{\partial y^2}$, $f_{xy} = \frac{\partial^2 f}{\partial x\,\partial y}$

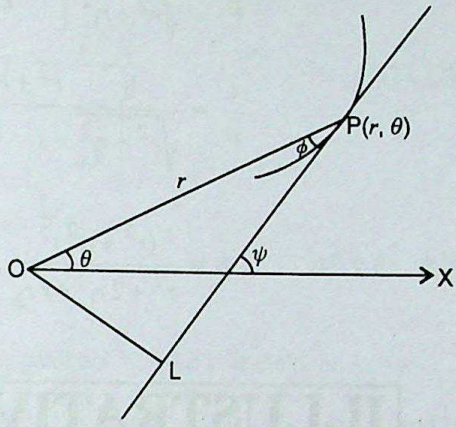
CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Art-8. Radius of Curvature for Polar Curves

Find radius of curvature at any point $P(r, \theta)$ of the curve $r = f(\theta)$.

Proof : Let $P(r, \theta)$ be any point on the curve $r = f(\theta)$. Let ψ be the angle which the tangent at P makes with positive direction of OX. Let ϕ be the angle between tangent and radius vector.



$$\therefore \qquad \psi = \theta + \phi$$

Differentiating both sides w.r.t. r, we get,

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}$$

$$= \frac{d\theta}{ds}\left[1 + \frac{d\phi}{ds} \cdot \frac{ds}{d\theta}\right]$$

$$= \frac{d\theta}{ds}\left[1 + \frac{d\phi}{d\theta}\right] \qquad \ldots(1)$$

From differential calculus, we have

$$\tan\phi = r\frac{d\theta}{dr}$$

$$\therefore \qquad \tan\phi = r \cdot \frac{1}{\frac{dr}{d\theta}} = \frac{r}{r_1}, \text{ where } r_1 = \frac{dr}{d\theta}$$

$$\therefore \qquad \phi = \tan^{-1}\frac{r}{r_1}$$

Differentiate w.r.t. θ, we get,

$$\frac{d\phi}{d\theta} = \frac{1}{1 + \frac{r^2}{r_1^2}} \cdot \frac{d}{d\theta}\left(\frac{r}{r_1}\right)$$

$$= \frac{r_1^2}{r^2 + r_1^2} \cdot \frac{r_1 \cdot r_1 - r r_2}{r_1^2} = \frac{r_1^2 - r r_2}{r^2 + r_1^2}$$

Also $$\frac{ds}{d\theta} = \sqrt{r^2 + r_1^2}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$\therefore$ from (1), we get,

$$\frac{1}{\rho} = \frac{1}{\sqrt{r^2 + r_1^2}}\left[1 + \frac{r_1^2 - r\,r_2}{r^2 + r_1^2}\right]$$

$$\therefore \quad \frac{1}{\rho} = \frac{1}{\sqrt{r^2 + r_1^2}} \frac{r^2 + r_1^2 + r_1^2 - r\,r_2}{r^2 + r_1^2}$$

$$\therefore \quad \rho = \frac{(r^2 + r_1^2)^{\frac{3}{2}}}{r^2 + 2\,r_1^2 - r\,r_2}.$$

ILLUSTRATIVE EXAMPLES

Example 1. Find the radius of curvature at the point (s, ψ) of the curve

$$s = a \sec\psi \tan\psi + a \log(\sec\psi + \tan\psi).$$

Sol. The equation of curve is

$$s = a \sec\psi \tan\psi + a \log(\sec\psi + \tan\psi)$$

$$\therefore \quad \frac{ds}{d\psi} = a(\sec\psi.\sec^2\psi + \tan\psi.\sec\psi.\tan\psi) + \frac{a}{\sec\psi + \tan\psi}(\sec\psi.\tan\psi + \sec^2\psi)$$

$$= a \sec\psi\,(\sec^2\psi + \tan^2\psi) + \frac{a}{(\sec\psi + \tan\psi)} \sec\psi\,(\sec\psi + \tan\psi)$$

$$= a \sec\psi\,(\sec^2\psi + \tan^2\psi) + a \sec\psi$$

$$= a \sec\psi\,[(1 + \tan^2\psi) + \sec^2\psi]$$

$$= a \sec\psi\,(\sec^2\psi + \sec^2\psi) = a \sec\psi\,(2 \sec^2\psi)$$

$$\frac{ds}{d\psi} = 2\,a \sec^3\psi$$

$\therefore$ radius of curvature $\rho = \dfrac{ds}{d\psi} = 2\,a \sec^3\psi$

Example 2. Prove that the curvature of a straight line is zero.

Sol. Let the equation of straight line be

$$a\,x + b\,y + c = 0 \quad \text{where} \quad a^2 + b^2 \neq 0.$$

Differentiating both sides w.r.t. x, we get

$$a + b\frac{dy}{dx} = 0 \qquad \text{or} \qquad \frac{dy}{dx} = -\frac{a}{b} \qquad \ldots(1)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

$$\therefore \quad \frac{d^2y}{dx^2} = 0 \qquad \ldots(2)$$

Now
$$\kappa = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} = \frac{0}{\left[1+\dfrac{a^2}{b^2}\right]^{\frac{3}{2}}} \qquad [\because \text{ of (1), (2)}]$$

$$= 0 \qquad [\because a^2 + b^2 \neq 0]$$

Hence the result.

Example 3. Find the radius of curvature of the parabola $y^2 = 4\,a\,x$ at the point (x, y).

Sol. The equation of the parabola is $y^2 = 4\,a\,x$ ...(1)

Differentiating both sides w.r.t x, we get,

$$2\,y\,\frac{dy}{dx} = 4\,a \quad \text{or} \quad \frac{dy}{dx} = \frac{2\,a}{y} \qquad \ldots(2)$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{2\,a}{y^2}\,\frac{dy}{dx} = -\frac{2\,a}{y^2} \times \frac{2\,a}{y} = -\frac{4\,a^2}{y^3} \qquad \ldots(3)$$

Now,
$$\rho = \frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$$

$$\therefore \quad \rho = \frac{\left(1+\dfrac{4\,a^2}{y^2}\right)^{\frac{3}{2}}}{\dfrac{-4\,a^2}{y^3}} \qquad [\because \text{ of (2) and (3)}]$$

$$= -\frac{(y^2+4\,a^2)^{\frac{3}{2}}}{4\,a^2} = -\frac{(4\,a\,x+4\,a^2)^{\frac{3}{2}}}{4\,a^2} \qquad [\because \text{ of (1)}]$$

$$= \frac{2}{\sqrt{a}}(x+a)^{\frac{3}{2}} \qquad \text{(in magnitude)}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 4. Find the radius of curvature at any point of the curve $y = c \cosh \frac{x}{c}$ and show that the curvature at any point varies inversely as the square of the ordinate.

Sol. The equation of curve is $y = c \cosh \frac{x}{c}$ ...(1)

$$\therefore \quad y_1 = c.\frac{1}{c}\sinh\frac{x}{c} = \sinh\frac{x}{c}$$

$$y_2 = \frac{1}{c}\cosh\frac{x}{c}$$

$$\therefore \quad \rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{\left(1+\sinh^2\frac{x}{c}\right)^{\frac{3}{2}}}{\frac{1}{c}\cosh\frac{x}{c}} = \frac{c\cosh^3\frac{x}{c}}{\cosh\frac{x}{c}}$$

$$\therefore \quad \rho = c\cosh^2\frac{x}{c} = c\left(\frac{y}{c}\right)^2 \qquad [\because \text{ of } (1)]$$

$$\Rightarrow \quad \frac{1}{\rho} = \frac{c}{y^2} \qquad \Rightarrow \quad \rho = \frac{y^2}{c}$$

$\Rightarrow$ curvature at any point $= \frac{c}{y^2}$, which varies inversely as the square of the ordinate at that point.

Example 5. Find the radius of curvature at any point of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Sol. The equation of curve is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ ...(1)

Differentiating both sides w.r.t. x, we get,

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = -\left(\frac{y}{x}\right)^{\frac{1}{3}} \qquad ...(2)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

$$\frac{d^2y}{dx^2} = -\frac{1}{3}\left(\frac{y}{x}\right)^{-\frac{2}{3}}\left[\frac{x.\frac{dy}{dx} - y.1}{x^2}\right]$$

$$= -\frac{1}{3}.\frac{x^{\frac{2}{3}}}{y^{\frac{2}{3}}}.\frac{1}{x^2}\left[-x\left(\frac{y}{x}\right)^{\frac{1}{3}} - y\right] \qquad [\because \text{ of } (2)]$$

$$= \frac{1}{3x^{\frac{4}{3}}y^{\frac{2}{3}}}\left[x^{\frac{2}{3}}y^{\frac{1}{3}} + y\right] = \frac{1}{3x^{\frac{4}{3}}y^{\frac{1}{3}}}\left[x^{\frac{2}{3}} + y^{\frac{2}{3}}\right]$$

$$= \frac{1}{3x^{\frac{4}{3}}y^{\frac{1}{3}}}.a^{\frac{2}{3}} \qquad [\because \text{ of } (1)]$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{a^{\frac{2}{3}}}{3x^{\frac{4}{3}}y^{\frac{1}{3}}} \qquad \ldots(3)$$

Now $$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$\therefore \quad \rho = \frac{\left[1 + \left(\frac{y}{x}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}}{\frac{a^{\frac{2}{3}}}{3x^{\frac{4}{3}}y^{\frac{1}{3}}}} \qquad [\because \text{ of } (2) \text{ and } (3)]$$

$$= \frac{\left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right)^{\frac{3}{2}}}{x} \times \frac{3x^{\frac{4}{3}}y^{\frac{1}{3}}}{a^{\frac{2}{3}}} = \frac{\left(a^{\frac{2}{3}}\right)^{\frac{3}{2}}}{x} \times \frac{3x^{\frac{4}{3}}y^{\frac{1}{3}}}{a^{\frac{2}{3}}}$$

$$= 3a^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}} = 3(axy)^{\frac{1}{3}}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 6. Prove that the radius of curvature at any point P(x, y) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{a^2b^2}{p^3}$, where p is the length of perpendicular from the centre of ellipse on the tangent at P.

Sol. The equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(1)

Differentiating both sides w.r.t. x, we get,

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\therefore \quad \frac{y}{b^2}\cdot\frac{dy}{dx}=-\frac{x}{a^2} \quad \Rightarrow \quad \frac{dy}{dx}=-\frac{b^2}{a^2}\left(\frac{x}{y}\right) \qquad ...(2)$$

$$\therefore \quad \frac{d^2y}{dx^2}=-\frac{b^2}{a^2}\left[\frac{y.1-x.\frac{dy}{dx}}{y^2}\right]=-\frac{b^2}{a^2y^2}\left[y+\frac{b^2}{a^2}\cdot\frac{x^2}{y}\right] \qquad [\because \text{ of (2)}]$$

$$=-\frac{b^2}{a^2y^2}\cdot\frac{b^2}{y}\left[\frac{y^2}{b^2}+\frac{x^2}{a^2}\right]=-\frac{b^4}{a^2y^3}(1) \qquad [\because \text{ of (1)}]$$

$$=-\frac{b^4}{a^2y^3}$$

Now $$\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=-\frac{\left(1+\frac{b^4x^2}{a^4y^2}\right)^{\frac{3}{2}}}{\frac{b^4}{a^2y^3}} \qquad [\because \text{ of (2), (3)}]$$

$$=\frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^6y^3}\times\frac{a^2y^3}{b^4} \qquad \text{(in magnitude)}$$

$$\therefore \quad \rho=\frac{1}{a^4b^4}(a^4y^2+b^4x^2)^{\frac{3}{2}} \qquad ...(4)$$

The equation of tangent at (X, Y) to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{is} \quad \frac{xX}{a^2}+\frac{yY}{b^2}-1=0$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education,Bantalab Jammu

$\therefore \quad p$ = length of perpendicular from (0, 0) on tangent

$$= \frac{|0+0-1|}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}}} = \frac{a^2b^2}{\sqrt{a^4y^2+b^4x^2}}$$

$$\therefore \quad \sqrt{a^4y^2+b^4x^2} = \frac{a^2b^2}{p} \quad \Rightarrow \quad (a^4y^2+b^4x^2)^{\frac{3}{2}} = \frac{a^6b^6}{p^3}$$

$$\therefore \quad \text{from (4)}, \rho = \frac{1}{a^4b^4} \times \frac{a^6b^6}{p^3} = \frac{a^2b^2}{p^3}$$

Example 7. Find the curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on the curve

$$x^3 + y^3 = 3\,a\,x\,y.$$

Sol. The equation of curve is $x^3 + y^3 = 3\,a\,x\,y$ ...(1)

Differentiating w.r.t. x, we get,

$$3\,x^2 + 3\,y^2\,\frac{dy}{dx} = 3\,a\left(x\frac{dy}{dx} + y\right)$$

$$\Rightarrow \quad (y^2 - a\,x)\,\frac{dy}{dx} = a\,y - x^2 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{a\,y - x^2}{y^2 - a\,x} \qquad \ldots(2)$$

At $\mathrm{P}\left(\frac{3a}{2}, \frac{3a}{2}\right), \frac{dy}{dx} = \dfrac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - \frac{3a^2}{2}} = -1$

$$\frac{d^2y}{dx^2} = \frac{(y^2 - a\,x)\left(a\frac{dy}{dx} - 2\,x\right) - (a\,y - x^2)\left(2\,y\frac{dy}{dx} - a\right)}{(y^2 - a\,x)^2}$$

At $\mathrm{P}\left(\frac{3a}{2}, \frac{3a}{2}\right)$,

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)(-a - 3a) - \left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)(-3a - a)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$= \frac{-3a^3 - 3a^3}{\frac{9a^4}{16}} = -6a^3 \times \frac{16}{9a^4} = -\frac{32}{3a}$$

$$\text{At P}, \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{(1+1)^{\frac{3}{2}}}{-\frac{32}{3a}} = -2\sqrt{2} \times \frac{3a}{32} = \frac{3a\sqrt{2}}{16} \text{ (in magnitude)}$$

Example 8. Find the radius of curvature at any point of the curve

$$x = a(t - \sin t),\ y = a(1 - \cos t).$$

Sol. The equations of the curve are

$$x = a(t - \sin t),\quad y = a(1 - \cos t)$$

$$\therefore \quad \frac{dx}{dt} = a(1 - \cos t),\quad \frac{dy}{dt} = a \sin t$$

$$\text{Now} \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a \sin t}{a(1 - \cos t)} = \frac{2a \sin \frac{t}{2} \cos \frac{t}{2}}{2a \sin^2 \frac{t}{2}} = \cot \frac{t}{2}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2} \operatorname{cosec}^2 \frac{t}{2} \cdot \frac{dt}{dx} = -\frac{1}{2} \operatorname{cosec}^2 \frac{t}{2} \cdot \frac{1}{a(1 - \cos t)}$$

$$= -\frac{1}{2} \operatorname{cosec}^2 \frac{t}{2} \cdot \frac{1}{2a \sin^2 \frac{t}{2}} = -\frac{1}{4a} \operatorname{cosec}^4 \frac{t}{2}$$

$$\text{Now} \quad \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left(1 + \cot^2 \frac{t}{2}\right)^{\frac{3}{2}}}{-\frac{1}{4a} \operatorname{cosec}^4 \frac{t}{2}} = -4a \frac{\left(\operatorname{cosec}^2 \frac{t}{2}\right)^{\frac{3}{2}}}{\operatorname{cosec}^4 \frac{t}{2}}$$

$$= -4a \frac{\operatorname{cosec}^3 \frac{t}{2}}{\operatorname{cosec}^4 \frac{t}{2}} = -\frac{4a}{\operatorname{cosec} \frac{t}{2}} = 4a \sin \frac{t}{2} \text{ (in magnitude)}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Example 9. Find the radius of curvature for the parabola $\frac{2a}{r} = 1 + \cos\theta$.

Sol. The equation of the curve is

$$\frac{2a}{r} = 1 + \cos\theta$$

or $$r = \frac{2a}{1+\cos\theta} \quad \text{or} \quad r = 2a(1+\cos\theta)^{-1}$$

$$\therefore \quad r_1 = \frac{dr}{d\theta} = \frac{-2a}{(1+\cos\theta)^2}.(-\sin\theta) = \frac{2a\sin\theta}{(1+\cos\theta)^2}$$

$$r_2 = 2a\left[\frac{(1+\cos\theta)^2.\cos\theta - \sin\theta.2(1+\cos\theta).(-\sin\theta)}{(1+\cos\theta)^4}\right]$$

$$= 2a\left[\frac{\cos\theta(1+\cos\theta) + 2\sin^2\theta}{(1+\cos\theta)^3}\right]$$

$$= 2a\left[\frac{\cos\theta + (\cos^2\theta + \sin^2\theta) + \sin^2\theta}{(1+\cos\theta)^3}\right] = 2a\left[\frac{1+\cos\theta+\sin^2\theta}{(1+\cos\theta)^3}\right]$$

Now $$\rho = \frac{(r^2 + r_1^2)^{\frac{3}{2}}}{r^2 + 2r_1^2 - r r_2}$$

$$\therefore \quad \rho = \frac{\left[\frac{4a^2}{(1+\cos\theta)^2} + \frac{4a^2\sin^2\theta}{(1+\cos\theta)^4}\right]^{\frac{3}{2}}}{\frac{4a^2}{(1+\cos\theta)^2} + 2.\frac{4a^2\sin^2\theta}{(1+\cos\theta)^4} - \frac{2a}{1+\cos\theta}.2a\frac{1+\cos\theta+\sin^2\theta}{(1+\cos\theta)^3}}$$

$$= \frac{\left[\frac{4a^2}{(1+\cos\theta)^4}\right]^{\frac{3}{2}}[(1+\cos\theta)^2 + \sin^2\theta]^{\frac{3}{2}}}{\left[\frac{4a^2}{(1+\cos\theta)^4}\right][(1+\cos\theta)^2 + 2\sin^2\theta - (1+\cos\theta+\sin^2\theta)]}$$

$$= \left[\frac{4a^2}{(1+\cos\theta)^4}\right]^{\frac{1}{2}}.\frac{[(1+\cos\theta)^2 + (1-\cos^2\theta)]^{\frac{3}{2}}}{(1+\cos\theta)^2 + (1-\cos^2\theta) - (1+\cos\theta)}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$= \frac{2a}{(1+\cos\theta)^2} \cdot \frac{(1+\cos\theta)^{\frac{3}{2}}[1+\cos\theta+1-\cos\theta]^{\frac{3}{2}}}{(1+\cos\theta)[1+\cos\theta+1-\cos\theta-1]}$$

$$= \frac{2a}{(1+\cos\theta)^{\frac{3}{2}}} \cdot \frac{2^{\frac{3}{2}}}{1} = 2a\left(\frac{2}{1+\cos\theta}\right)^{\frac{3}{2}} = \frac{2}{\sqrt{a}}\left(\frac{2a}{1+\cos\theta}\right)^{\frac{3}{2}} = \frac{2}{\sqrt{a}} \cdot r^{\frac{3}{2}}$$

$$\therefore \qquad \rho = 2 \cdot \sqrt{\frac{r^3}{a}}$$

EXERCISE 7 (*a*)

1. Find the radius of curvature at any point (s, ψ) on the following curves :

   (*i*) $s = c \tan\psi$ (*ii*) $s = 8\,a \sin^2 \frac{\psi}{6}$

   (*iii*) $s = c \log \sec\psi$ (*iv*) $s = a \log \tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right)$

2. Find the radius of curvature at the point (x, y) on the curve $y = a \log \sec\left(\frac{x}{a}\right)$.

3. Find the radius of curvature at any point on the curve $xy = c^2$.

4. Find the radius of curvature at any point of the following curve :

   (*i*) $ay^2 = x^3$ (*ii*) $y = \log \sin x$

5. Show that $\frac{3\sqrt{3}}{2}$ is the least value of $|\rho|$ for $y = \log x$.

6. Find the points on the parabola $y^2 = 8x$ at which the radius of curvature is $7\frac{13}{16}$. (J.U. 2016)

7. Prove that if ρ be the radius of curvature at any point P on the parabola $y^2 = 4ax$ and S be its focus, then ρ^2 varies as $(SP)^3$.

8. Find the radius of curvature at any point of the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ and show that it is equal to the length of the normal at the same point and that it varies as the square of the ordinate.

9. Show that the radius of curvature at the point $(-2a, 2a)$ on the curve $x^2 y = a(x^2 + y^2)$ is $-2a$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

10. Find the radius of curvature for $\sqrt{\frac{x}{a}} - \sqrt{\frac{y}{b}} = 1$ at the points where it touches the co-ordinate axes.

11. Show that the ellipse $x = a \cos t, y = b \sin t, a > 0, b > 0$ has its largest curvature on its major axis and its smallest curvature on its minor axis.

12. Find the radius of curvature at any point of the cycloid

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta).$$

13. Find the radius of curvature at any point of the curve

(*i*) $x = a t^2, y = 2 a t$ (*ii*) $x = a \sec\theta, y = b \tan\theta$

(*iii*) $x = a \cos\theta, y = b \sin\theta$ (*iv*) $x = a \cos^3\theta, y = a \sin^3\theta$

(*v*) $x = a \sin 2\theta (1 + \cos 2\theta), y = a \cos 2\theta (1 - \cos 2\theta)$

(*vi*) $x = a(\cos t + t \sin t), y = a(\sin t - t \cos t)$

14. Show that the radius of curvature at each point of the curve

$$x = a\left(\cos t + \log\tan\frac{t}{2}\right), y = a \sin t$$

is inversely proportional to the length of the normal intercepted between the point on the curve and the x-axis.

15. Prove that the radius of curvature at any point of the curve

$$x = t - c \sinh\frac{t}{c} \cosh\frac{t}{c}, y = 2c \cosh\frac{t}{c}$$

is $-2c \cosh^2\frac{t}{c} \sinh\frac{t}{c}$ where t is the parameter.

16. If ρ_1, ρ_2 be the radii of curvature at the extremities of any chord of the cardioid $r = a(1 + \cos\theta)$ which passes through the pole, show that $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$.

17. Find the radius of curvature at the point (r, θ) on each of the curve :

(*i*) $r = a(1 - \cos\theta)$ (*ii*) $r^n = a^n \cos n\theta$.

18. Show that at each point of the lemniscate $r^2 = a^2 \cos 2\theta$, the curvature is inversely proportional to the radius vector at that point.

19. Show that at the point of intersection of the curves $r = a\theta$ and $r\theta = a$, the curvatures are in the ratio 3 : 1. $(0 < \theta < 2\pi)$

20. For the cardioid, show that $\frac{\rho^2}{r}$ is constant.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

21. If ρ_1, ρ_2 are the radii of curvature at the extremities of a focal chord of a parabola whose semi-latus rectum is l, prove that $(\rho_1)^{-\frac{2}{3}} + (\rho_2)^{-\frac{2}{3}} = (l)^{-\frac{2}{3}}$

22. In an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, show that the radius of curvature at the end of the major axis is equal to the semi-latus rectum of the ellipse.

23. Prove that for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\rho = \frac{CD^3}{ab}$, where CD is the semi conjugate diameter to CP.

24. If ρ_1 and ρ_2 be the radii of curvature at the extremities of the conjugate diameter of the ellipse, prove that $\left(\rho_1^{\frac{2}{3}} + \rho_2^{\frac{2}{3}}\right)(ab)^{\frac{2}{3}} = a^2 + b^2$.

25. The tangents at two points P, Q on the cycloid $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$ are at right angles, show that if ρ_1, ρ_2 be the radii of curvature at these points, then $\rho_1^2 + \rho_2^2 = 16a^2$.

26. Prove that for any curve, $\frac{r}{\rho} = \sin\phi\left(1 + \frac{d\phi}{d\theta}\right)$ where ρ is the radius of curvature and $\tan\phi = r\frac{d\theta}{dr}$.

27. Find the point on the curve $y = e^x$ at which the curvature is maximum, and show that the tangent at this point forms with the axes of co-ordinates a triangle whose sides are in the ratio $1 : \sqrt{2} : \sqrt{3}$.

28. Find ρ for the curve $s^2 = 8ay$.

29. Prove that for the curve $s = ce^{\frac{x}{c}}$, $c\rho = s\sqrt{s^2 - c^2}$.

30. Find the radius of curvature for the catenary

$$x = c\log\left(s + \sqrt{c^2 + s^2}\right), y = \sqrt{c^2 + s^2}$$

31. Show that for the curve $s = f(x)$, $\rho = \dfrac{\left(\frac{ds}{dx}\right)^2\sqrt{\left(\frac{ds}{dx}\right)^2 - 1}}{\frac{d^2s}{dx^2}}$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

ANSWERS

1. (*i*) $c \sec^2 \psi$ (*ii*) $\frac{4a}{3} \sin \frac{\psi}{3}$ (*iii*) $c \tan \psi$ (*iv*) $a \sec \psi$

2. $a \sec \frac{x}{a}$ 3. $\frac{(x^4 + c^4)^{\frac{3}{2}}}{2c^2 x^3}$

4. (*i*) $\frac{\sqrt{x}\,(4a + 9x)^{\frac{3}{2}}}{6a}$ (*ii*) $\operatorname{cosec} x$

6. $\left(\frac{9}{8}, 3\right), \left(\frac{9}{8}, -3\right)$ 8. $a \cosh^2 \frac{x}{a}$ 10. $\frac{2a^2}{b}, \frac{2(b^2 + 4a^2)^{\frac{3}{2}}}{ab}$

12. $4a \cos \frac{\theta}{2}$

13. (*i*) $2a\,(1 + t^2)^{\frac{3}{2}}$ (*ii*) $\frac{1}{ab}(a^2 \tan^2 \theta + b^2 \sec^2 \theta)^{\frac{3}{2}}$

(*iii*) $\frac{1}{ab}(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{\frac{3}{2}}$ (*iv*) $3a \sin \theta \cos \theta$

(*v*) $4a \cos 3\theta$ (*vi*) at

27. $\left(-\frac{1}{2} \log 2, \frac{1}{\sqrt{2}}\right)$ 28. $\sqrt{8a(2a - y)}$ 30. $\frac{c^2 + s^2}{c}$

Art-9. Radius of Curvature at the Origin

We know that the radius of curvature at the origin for curve $y = f(x)$ can be found by putting $x = 0, y = 0$ in the value of ρ obtained. Now we give direct methods for it.

(*i*) **Method of expansion :** By Maclaurin's expansion, we have

$$y = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots\ldots\ldots$$

Since the curve passes through the origin

$\therefore \quad f(0) = 0$

$$\therefore \quad y = x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots\ldots\ldots\ldots$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

or $$y = px + q\frac{x^2}{\lfloor 2} + \ldots\ldots\ldots\ldots$$

where $p = f'(0) = \left(\frac{dy}{dx}\right)_{(0,0)}$ and $q = f''(0) = \left(\frac{d^2y}{dx^2}\right)_{(0,0)}$

Now ρ (at the origin) $= \frac{(1+p^2)^{\frac{3}{2}}}{q}$

Note. Rule to find the radius of curvature at the origin

(*a*) Put $y = px + q\frac{x^2}{\lfloor 2} + \ldots\ldots\ldots\ldots$ in equation of curve

(*b*) Equate the coefficients of like powers of x on both sides and find p, q.

(*c*) ρ (at the origin) $= \frac{(1+p^2)^{\frac{3}{2}}}{q}$

(*ii*) **Newton's method**

If the curve passes through the origin and the axis of x is a tangent to the curve at the origin, then in such a case, we have at the origin

$$x = 0, y = 0 \text{ and } \frac{dy}{dx} = 0 \text{ } i.e., p = 0$$

$\therefore$ by Maclaurin's expansion as above, we have

$$y = 0 + 0.x + q.\frac{x^2}{\lfloor 2} + \ldots\ldots\ldots\ldots$$

or $$\frac{2y}{x^2} = q + \text{terms of } x$$

Taking limits as $x \to 0$ and consequently also $y \to 0$, we have

$$\underset{x \to 0}{\text{Lt}} \frac{2y}{x^2} = q$$

Also $$\rho = \frac{(1+p^2)^{\frac{3}{2}}}{q} = \frac{(1+0)^{\frac{3}{2}}}{q} = \frac{1}{q}$$

$\therefore$ $$\rho = \underset{x \to 0}{\text{Lt}} \frac{x^2}{2y}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Similarly, if the curve passes through the origin and the axis of y is the tangent there, then the value of ρ at the origin is

$$\rho = \underset{x \to 0}{\text{Lt}} \frac{y^2}{2x}$$

Note. Rule to find the radius of curvature at the origin of a curve

(*a*) Obtain the equation of tangent (tangents) at the origin

(*b*) (*i*) If x-axis is the tangent at origin, then ρ (at origin) $= \underset{x \to 0}{\text{Lt}} \dfrac{x^2}{2y}$

(*ii*) if y-axis is the tangent at origin, then ρ (at origin) $= \underset{x \to 0}{\text{Lt}} \dfrac{y^2}{2x}$

(*iii*) If neither x-axis nor y-axis is the tangent at origin, then put

$$y = px + q\frac{x^2}{\lfloor 2} + \ldots\ldots\ldots\ldots$$

and proceed as above.

ILLUSTRATIVE EXAMPLES

Example 1. Find the radii of curvature at the origin of the curve

$$y^2 - 3xy + 2x^2 - x^3 + y^4 = 0$$

Sol. The equation of curve is $y^2 - 3xy + 2x^2 - x^3 + y^4 = 0$ ...(1)

Clearly (0, 0) lies on (1).

Equating to zero the lowest degree terms, we get,

$$y^2 - 3xy + 2x^2 = 0$$

or $(y - x)(y - 2x) = 0 \quad \Rightarrow y = x, y = 2x$

Here, neither x-axis nor y-axis is the tangent at origin

$\therefore$ putting $y = px + q\dfrac{x^2}{\lfloor 2} + \ldots\ldots\ldots$ in (1), we get,

$$\left(px + q\frac{x^2}{2} + \ldots\ldots\right)^2 - 3x\left(px + q\frac{x^2}{2} + \ldots\ldots\right) + 2x^2 - x^3 + \left(px + q\frac{x^2}{2} + \ldots\ldots\right)^4 = 0$$

$$\Rightarrow \quad (p^2 - 3p + 2)x^2 + \left(pq - 3\frac{q}{2} - 1\right)x^3 + \ldots\ldots\ldots = 0 \qquad \ldots(2)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Equating coefficients of x^2 in (2), we get,

$$p^2 - 3p + 2 = 0 \text{ or } (p-1)(p-2) = 0 \quad \Rightarrow \quad p = 1, 2$$

Equating coefficients of x^3 in (2), we get,

$$pq - 3\frac{q}{2} - 1 = 0$$

When $\quad p = 1,\ q - 3\dfrac{q}{2} - 1 = 0 \quad \Rightarrow \quad q = -2$

When $\quad p = 2,\ 2q - 3\dfrac{q}{2} - 1 = 0 \quad \Rightarrow \quad q = 2$

When $\quad p = 1, q = -2$

$$\rho \text{ (at origin)} = \frac{(1+p^2)^{\frac{3}{2}}}{q} = \frac{(1+1)^{\frac{3}{2}}}{-2} = \sqrt{2} \qquad \text{(in magnitude)}$$

When $p = 2,\ q = 2$

$$\rho \text{ (at origin)} = \frac{(1+p^2)^{\frac{3}{2}}}{q} = \frac{(1+4)^{\frac{3}{2}}}{2} = \frac{5\sqrt{5}}{2}$$

Example 2. Show that the radii of curvature of the curve

$$y^2 = x^2\left(\frac{a+x}{a-x}\right) \text{ at the origin are } \pm\ a\sqrt{2}.$$

Sol. The equation of given curve is

$$y^2 = x^2\left(\frac{a+x}{a-x}\right) \qquad \text{...(1)}$$

or $\quad ay^2 - xy^2 = ax^2 + x^3$

or $\quad x^3 + xy^2 + ax^2 - ay^2 = 0$

Clearly (0, 0) lies on it.

Equating to zero the lowest degree terms, we get

$$ax^2 - ay^2 = 0 \quad \text{or} \quad y^2 = x^2$$

$\therefore \qquad y = x,\ y = -x$

Hence neither x-axis nor y-axis is the tangent at the origin.

From (1), $\qquad y = \pm\ x\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{2}}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\therefore \qquad y = \pm\, x\left(\frac{1+\frac{x}{a}}{1-\frac{x}{a}}\right)^{\frac{1}{2}} = \pm\, x\left(1+\frac{x}{a}\right)^{\frac{1}{2}}\left(1-\frac{x}{a}\right)^{-\frac{1}{2}}$$

or
$$y = \pm\, x\left(1+\frac{x}{2a}+\ldots\ldots\right)\left(1+\frac{x}{2a}+\ldots\ldots\right)$$

[By expanding with Binomial Theorem]

$$\therefore \qquad y = \pm\, x\left(1+\frac{x}{a}+\ldots\ldots\right)$$

or
$$y = \pm\, x \pm \frac{x^2}{a} \pm \ldots\ldots$$

Comparing it with $y = px + \frac{qx^2}{\lfloor 2} + \ldots\ldots$, we get

$$p = \pm 1,\ \frac{q}{2} = \pm\frac{1}{a} \quad i.e.\ \ p = \pm 1,\ q = \pm\frac{2}{a}$$

When $p = 1,\ q = \frac{2}{a}$, we have

$$\kappa = \frac{q}{(1+p^2)^{\frac{3}{2}}} = \frac{\frac{2}{a}}{(1+1)^{\frac{3}{2}}} = \frac{2}{a}\times\frac{1}{2^{\frac{3}{2}}} = \frac{2}{a}\times\frac{1}{2\sqrt{2}} = \frac{1}{a\sqrt{2}}$$

When $p = -1,\ q = -\frac{2}{a}$, we have

$$\kappa = \frac{q}{(1+p^2)^{\frac{3}{2}}} = \frac{-\frac{2}{a}}{(1+1)^{\frac{3}{2}}} = -\frac{1}{a\sqrt{2}}$$

$\therefore$ radii of curvature are $\pm\, a\sqrt{2}$.

Example 3. Find the radius of curvature for the curve $3x^2 + 4x^3 - 12y = 0$

Sol. The equation of curve is $3x^2 + 4x^3 - 12y = 0$ ...(1)

Clearly (0, 0) lies on (1)

The tangent at (0, 0) is given by

$-12y = 0$ *i.e.*, $y = 0$ *i.e.*, the axis of x

$$\therefore \quad \rho\ (\text{at origin}) = \underset{x\to 0}{\text{Lt}}\ \frac{x^2}{2y} \qquad \ldots(2)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Dividing both sides of (1) by $2y$, we get,

$$3\frac{x^2}{2y}+4x\left(\frac{x^2}{2y}\right)-6=0$$

Taking limits as $x \to 0$, we get,

$$3\rho+0\cdot\rho-6=0 \qquad [\because \text{ of } (2)]$$

$$\Rightarrow \quad \rho=2$$

EXERCISE 7 (*b*)

1. Obtain the radii of curvature at origin for the curve $a(y^2-x^2)=x^3$.
2. Find the radius of curvature at the origin of the curve
$$x^3-2x^2y+3xy^2-4y^3+5x^2-6xy+7y^2-8y=0.$$
3. Apply Newton's methods to find the radius of curvature at the origin for the cycloid $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$.
4. Find radii of curvature at the origin of the curve $x=1-t^2$, $y=t-t^3$.

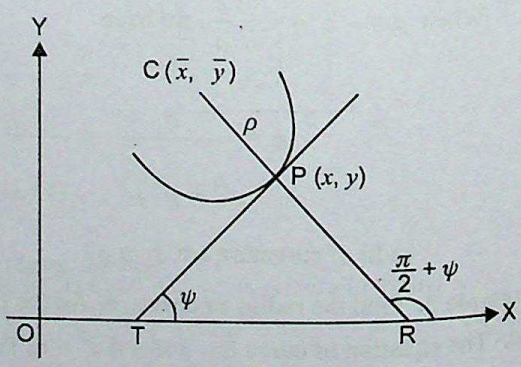
ANSWERS

1. $2\sqrt{2}\,a$ 2. $\frac{4}{5}$ 3. $4a$ 4. $2\sqrt{2}$

Art-10. Centre of Curvature

Find the co-ordinates of the centre of curvature for any point P (x, y) of the curve $y=f(x)$.

Proof : Let P(x, y) be any point on the given curve $y=f(x)$. Let the centre of curvature for P be the point $C(\bar{x},\bar{y})$ which lies on the normal at a distance ρ from it.



Let the positive direction of the tangent make an angle ψ with x-axis so that the positive direction of the normal makes an angle $\left(\psi+\frac{\pi}{2}\right)$ with x-axis. So the equation of normal to the curve at P is

$$\frac{X-x}{\cos\left(\frac{\pi}{2}+\psi\right)}=\frac{Y-y}{\sin\left(\frac{\pi}{2}+\psi\right)} \quad \text{or} \quad \frac{X-x}{-\sin\psi}=\frac{Y-y}{\cos\psi}=r$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

where X, Y are the current co-ordinates of any point on the normal and r is the distance of the variable point (X, Y) from (x, y)

Now any point on this normal is $(x - r\sin\psi, y + r\cos\psi)$

It will be the centre of curvature *i.e.*, the point $C(\bar{x}, \bar{y})$ if $r = PC = \rho$. Hence we have

$$\bar{x} = x - \rho\sin\psi \text{ and } \bar{y} = y + \rho\cos\psi \qquad \ldots(1)$$

But we know that at any point (x, y)

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}, \quad \sin\psi = \frac{\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

$$\cos\psi = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} \qquad \left[\because \tan\psi = \frac{dy}{dx}\right]$$

Hence, from (1), we obtain

$$\bar{x} = x - \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \cdot \frac{\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = x - \frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

and $$\bar{y} = y + \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \cdot \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$

$\therefore$ the centre of curvature for the point (x, y) is $\left(x - \frac{y_1(1+y_1^2)}{y_2}, y + \frac{1+y_1^2}{y_2}\right)$

Cor. The equation of the circle of curvature at P (x, y) is $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$ where (x, y) are the co-ordinates of the centre of curvature corresponding to P and ρ is the radius of curvature at P.

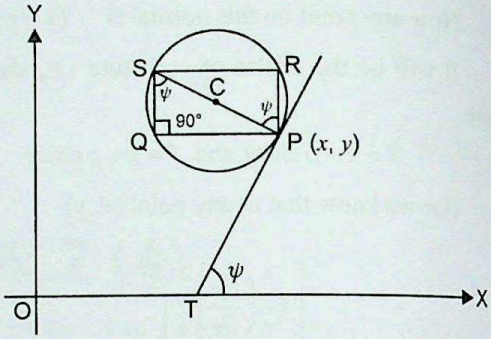
Note. The locus of the centre of curvature of a curve is called its **evolute** and the curve itself is called **involute**.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Art-11. Find the lengths of the chords of curvature parallel to x-axis and y-axis.

Proof : Let P(x, y) be any point on a curve and PT be the tangent there, which makes an angle ψ with the positive direction of the x-axis. Let C be the centre of curvature corresponding to P. We draw the circle of curvature with C as centre and PC = ρ as radius. PQ and PR are chords of this circle passing through P and parallel to x-axis and y-axis respectively.



We have to determine their lengths

Clearly $\angle PSQ = \psi = \angle SPR$

(*i*) length of chord of curvature parallel to x-axis

$$= PQ = PS \sin \angle PSQ = 2\rho \sin \psi$$

$$= 2\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} \cdot \frac{y_1}{\sqrt{1+y_1^2}} = \frac{2y_1(1+y_1^2)}{y_2}$$

(*ii*) length of chord of curvature parallel to y-axis

$$= PR = PS \cos \angle SPR = 2\rho \cos \psi$$

$$= 2\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} \cdot \frac{1}{\sqrt{1+y_1^2}} = \frac{2(1+y_1^2)}{y_2}$$

ILLUSTRATIVE EXAMPLES

Example 1. Find the centre of curvature at any point (x, y) of the parabola $y^2 = 4ax$. Also find its evolute.

Sol. The equation of parabola is $y^2 = 4ax$

Differentiating both sides w.r.t. x, we get,

$$2y\frac{dy}{dx} = 4a$$

or $$\frac{dy}{dx} = \frac{2a}{y} \quad \text{...(1)}$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{2a}{y^2} \cdot \frac{dy}{dx} = -\frac{4a^2}{y^3} \quad \text{...(2)}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

If $(\bar{x}, \bar{y})$ be the co-ordinates of the centre of curvature, then

$$\bar{x} = x - \frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

$$= x - \frac{\frac{2a}{y}\left[1+\frac{4a^2}{y^2}\right]}{-\frac{4a^2}{y^3}} \qquad [\because \text{ of (1) and (2)}]$$

$$= x + \frac{y^2+4a^2}{2a}$$

$$= \frac{2ax+4ax+4a^2}{2a}$$

$$\therefore \quad \bar{x} = 3x + 2a \qquad \ldots(3)$$

and $$\bar{y} = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = y + \frac{1+\frac{4a^2}{y^2}}{-\frac{4a^2}{y^3}} \qquad [\because \text{ of (1) and (2)}]$$

$$= y - \frac{y(y^2+4a^2)}{4a^2} = -\frac{y^3}{4a^2} = -\frac{(4ax)^{\frac{3}{2}}}{4a^2} \qquad [\because y^2 = 4ax]$$

$$\therefore \quad \bar{y} = -2a^{-\frac{1}{2}}x^{\frac{3}{2}} \qquad \ldots(4)$$

$\therefore$ centre of curvature at (x, y) is the point $\left(3x+2a, -2a^{-\frac{1}{2}}x^{\frac{3}{2}}\right)$

In order to find evolute, we eliminate x from (3) and (4).

Squaring (4), $\bar{y}^2 = \frac{4x^3}{a}$

$$\therefore \quad \bar{y}^2 = \frac{4}{a}\left(\frac{\bar{x}-2a}{3}\right)^3 \qquad [\because \text{ of (3)}]$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

or $\quad 27\, a\, \bar{y}^2 = 4\,(\bar{x} - 2\,a)^3$

$\therefore$ locus of $(\bar{x}, \bar{y})$ is $27\, a\, y^2 = 4\,(x - 2\,a)^3$

which is the required evolute.

Example 2. Find the co-ordinates of the centre of curvature of any point of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Hence obtain its evolute.

Sol. The equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Its parametric equations are $x = a \sec\theta, y = b \tan\theta$

$$\therefore \quad \frac{dx}{d\theta} = a \sec\theta \tan\theta\,, \quad \frac{dy}{d\theta} = b \sec^2\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b \sec^2\theta}{a \sec\theta \tan\theta} = \frac{b}{a} \operatorname{cosec}\theta$$

$$\frac{d^2y}{dx^2} = -\frac{b}{a} \operatorname{cosec}\theta \cot\theta . \frac{d\theta}{dx}$$

$$= -\frac{b}{a} \operatorname{cosec}\theta \cot\theta \times \frac{1}{a \sec\theta \tan\theta} = -\frac{b}{a^2} \cot^3\theta$$

Let $(\bar{x}, \bar{y})$ be the centre of curvature

$$\therefore \quad \bar{x} = x - \frac{y_1\left(1 + y_1^2\right)}{y_2}$$

$$= a \sec\theta - \frac{\frac{b}{a} \operatorname{cosec}\theta \left(1 + \frac{b^2}{a^2} \operatorname{cosec}^2\theta\right)}{-\frac{b}{a^2} \cot^3\theta}$$

$$= a \sec\theta + a\, \frac{\sin^2\theta}{\cos^3\theta} \left(1 + \frac{b^2}{a^2 \sin^2\theta}\right)$$

$$= \frac{a}{\cos\theta} + \frac{a \sin^2\theta}{\cos^3\theta} + \frac{b^2}{a \cos^3\theta}$$

$$= \frac{a^2 \cos^2\theta + a^2 \sin^2\theta + b^2}{a \cos^3\theta} = \frac{a^2 + b^2}{a \cos^3\theta}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\therefore \quad \bar{x} = \frac{a^2+b^2}{a}\sec^3\theta \qquad \ldots(1)$$

$$\bar{y} = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = b\tan\theta + \frac{1+\frac{b^2}{a^2}\operatorname{cosec}^2\theta}{-\frac{b}{a^2}\cot^3\theta}$$

$$= b\tan\theta - \frac{a^2}{b}\tan^3\theta - b\tan\theta\sec^2\theta$$

$$= -\,b\tan\theta(\sec^2\theta - 1) - \frac{a^2}{b}\tan^3\theta$$

$$= -b\tan^3\theta - \frac{a^2}{b}\tan^3\theta$$

$$\therefore \quad \bar{y} = -\frac{a^2+b^2}{b}\tan^3\theta \qquad \ldots(2)$$

$\therefore$ centre of curvature is $\left(\frac{a^2+b^2}{a}\sec^3\theta\,,\, -\frac{a^2+b^2}{b}\tan^3\theta\right)$

For evolute, we are to eliminate θ from (1) and (2)

From (1), $a\bar{x} = (a^2+b^2)\sec^3\theta$

$$\therefore \quad (a\bar{x})^{\frac{2}{3}} = (a^2+b^2)^{\frac{2}{3}}\sec^2\theta \qquad \ldots(3)$$

From (2), $b\,\bar{y} = -(a^2+b^2)\tan^3\theta$

$$\therefore \quad (b\bar{y})^{\frac{2}{3}} = (a^2+b^2)^{\frac{2}{3}}\tan^2\theta \qquad \ldots(4)$$

Subtracting (4) from (3), we get,

$$(a\bar{x})^{\frac{2}{3}} - (b\bar{y})^{\frac{2}{3}} = (a^2+b^2)^{\frac{2}{3}}(\sec^2\theta - \tan^2\theta)$$

$\therefore$ locus of $(\bar{x}, \bar{y})$ is $(ax)^{\frac{2}{3}} - (by)^{\frac{2}{3}} = (a^2+b^2)^{\frac{2}{3}}$

which is the required evolute.

Example 3. Find the circle of curvature at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$ of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$.

Sol. The equation of curve is $\sqrt{x}+\sqrt{y}=\sqrt{a}$ $\qquad \ldots(1)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Jammu

Differentiating both sides of (1) w.r.t x.

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0 \quad \Rightarrow \quad \frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}$$

$$\frac{d^2y}{dx^2}=-\frac{\sqrt{x}\cdot\frac{1}{2\sqrt{y}}\frac{dy}{dx}-\sqrt{y}\cdot\frac{1}{2\sqrt{x}}}{x}$$

$$=-\frac{1}{x}\left[-\frac{\sqrt{x}}{2\sqrt{y}}\times\frac{\sqrt{y}}{\sqrt{x}}-\frac{\sqrt{y}}{2\sqrt{x}}\right]$$

$$=\frac{1}{2x}\left[1+\frac{\sqrt{y}}{\sqrt{x}}\right]=\frac{1}{2x}\left[\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}}\right]=\frac{1}{2x}\left(\frac{\sqrt{a}}{\sqrt{x}}\right) \qquad [\because \text{ of (1)}]$$

$$=\frac{\sqrt{a}}{2x^{\frac{3}{2}}}$$

At P $\left(\frac{a}{4},\frac{a}{4}\right)$, $\quad y_1=-1, \quad y_2=\dfrac{\sqrt{a}}{2\left(\frac{a}{4}\right)^{\frac{3}{2}}}=\dfrac{4}{a}$

$$\therefore \quad \rho=\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}=\frac{(1+1)^{\frac{3}{2}}}{\frac{4}{a}}=2\sqrt{2}\times\frac{a}{4}=\frac{a}{\sqrt{2}}$$

Let $(\bar{x},\bar{y})$ be the centre of curvature

$$\therefore \quad \bar{x}=x-\frac{y_1(1+y_1^2)}{y_2}=\frac{a}{4}-\frac{(-1)(1+1)}{\frac{4}{a}}=\frac{a}{4}+\frac{a}{2}=\frac{3a}{4}$$

$$\bar{y}=y+\frac{1+y_1^2}{y_2}=\frac{a}{4}+\frac{(1+1)}{\frac{4}{a}}=\frac{3a}{4}$$

$\therefore$ centre of curvature is $\left(\frac{3a}{4},\frac{3a}{4}\right)$

$\therefore$ equation of circle of curvature at P $\left(\frac{a}{4},\frac{a}{4}\right)$ is

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\left(x-\frac{3a}{4}\right)^2+\left(y-\frac{3a}{4}\right)^2=\left(\frac{a}{\sqrt{2}}\right)^2$$

or $$\left(x-\frac{3a}{4}\right)^2+\left(y-\frac{3a}{4}\right)^2=\frac{a^2}{2}$$

Example 4. In the curve $y = a \log \sec\left(\frac{x}{a}\right)$, prove that the chord of curvature parallel to the axis of y is of constant length.

Sol. Equation of the given curve is $y = a \log \sec\left(\frac{x}{a}\right)$

Differentiating it w.r.t. x, we get,

$$\frac{dy}{dx} = a\,\frac{\sec\frac{x}{a}\tan\frac{x}{a}}{\sec\frac{x}{a}}\cdot\frac{1}{a} = \tan\frac{x}{a} \qquad \ldots(1)$$

But $\frac{dy}{dx} = \tan\psi$

$\therefore$ from (1), we get, $\tan\frac{x}{a} = \tan\psi$

$$\Rightarrow \quad \psi = \frac{x}{a} \qquad \ldots(2)$$

Now differentiating (1) again w.r.t x we get,

$$\frac{d^2y}{dx^2} = \frac{1}{a}\sec^2\frac{x}{a} \qquad \ldots(3)$$

So ρ, the radius of curvature at any point (x, y) of the curve, is given by

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\tan^2\frac{x}{a}\right]^{\frac{3}{2}}}{\frac{1}{a}\sec^2\left(\frac{x}{a}\right)} = a\sec\frac{x}{a} \qquad \ldots(4)$$

Hence the chord of curvature parallel to the axis of y

$$= 2\rho\cos\psi = 2\left[a\sec\frac{x}{a}\right]\left[\cos\frac{x}{a}\right] \qquad [\because \text{ of (2) and (4)}]$$

$= 2a$, which is constant.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

EXERCISE 7 (c)

1. Find the centre of curvature at any point (x, y) of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Also find evolute of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

2. Show that the evolute of $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ is $(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2a^{\frac{2}{3}}$

3. Show that the circle of curvature at the origin for the curve
$x+y=ax^2+by^2+cx^3$ is $(a+b)(x^2+y^2)=2(x+y)$.

4. If C_x, C_y be the chords of curvature parallel to axes of x and y respectively, at any point of the curve $y=ae^{\frac{x}{a}}$, then prove that $\frac{1}{C_x^2}+\frac{1}{C_y^2}=\frac{1}{2aC_x}$.

5. If C_x, C_y be the lengths of chords of curvature parallel to the co-ordinate axes at any point of the curve $y=c\cosh\frac{x}{c}$, then prove that $4c^2(C_x^2+C_y^2)=C_y^4$.

ANSWERS

1. $\bar{x}=\frac{(a^2-b^2)x^3}{a^4}$, $\bar{y}=\frac{y^3}{b^4}(b^2-a^2)$; $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

8

ASYMPTOTES

Art-1. Introduction

If the equation $f(x, y) = 0$ of the curve, when solved for y in terms of x, gives $y = \phi_1(x)$, $y = \phi_2(x)$, ..., $y = \phi_n(x)$, we say that the curve $f(x, y) = 0$ has n branches. Out of these n branches, the curve may have a branch or branches extending beyond the finite region. Such branch or branches are called infinite. If P(x, y) is a point on such a branch so that atleast one of x and y tends to ∞ or $-\infty$, then P is said to tend to infinity and we write it as $P \to \infty$.

Art-2. Asymptote

A straight line l is called an asymptote of an infinite branch of a curve iff the perpendicular distance of a point P on that branch from the straight line l tends to zero as P moves to infinity along the branch.

Rectangular Asymptote

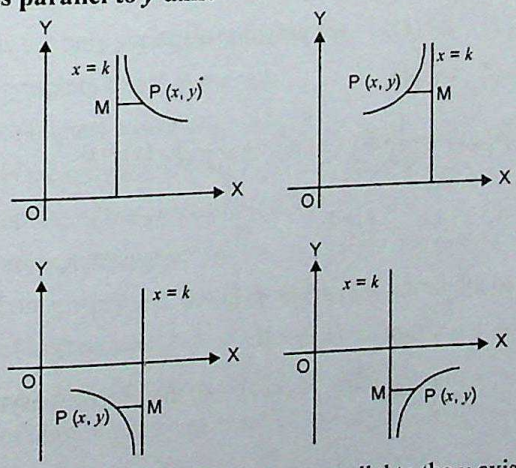
If an asymptote to a curve is either parallel to x-axis or parallel to y-axis, then it is called a **rectangular asymptote.** An asymptote parallel to x-axis is usually called **horizontal asymptote** and an asymptote parallel to y-axis is called a **vertical asymptote.**

Oblique asymptote

An asymptote, which is neither parallel to x-axis nor parallel to y-axis is called an oblique asymptote.

Art-3. Asymptotes Parallel to the AXES

(*a*) Asymptotes parallel to *y*-axis.



Let $x = k$ be an asymptote of the curve $y = f(x)$ parallel to the y-axis. We determine k.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Let P (x, y) be any point on the curve. Draw PM perpendicular on $x = k$.

$\therefore \quad |PM| = |x - k|$

Let P (x, y) move to infinity along the curve.

$\therefore$ y alone tends to $+\infty$ or $-\infty$

$\therefore \quad \underset{x \to k}{\text{Lt}} \; y = \pm\infty$

Equivalently $k = \underset{y \to \pm\infty}{\text{Lt}} \; x$

Thus k is determined.

Case I. Let the equation of the curve be of the form $y = \dfrac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials in x without any common factor.

Let $x = k_1, k_2, \ldots, k_r$ be the roots of $D(x) = 0$

$\therefore$ as $x \to k_1$, $D(x) \to 0$

$\therefore$ $y \to \infty$ as $x \to k_1$

$\therefore$ $x = k_1$ is a vertical asymptote of the curve $y = f(x)$

Similarly $x = k_2$, $x = k_3$, ... are the vertical asymptotes of the curve $y = \dfrac{N(x)}{D(x)}$

By factor theorem, $(x - k_1)$, $(x - k_2)$, ..., $(x - k_n)$ are the linear factors of $D(x)$.

Thus, linear factors of $D(x)$ equated to zero determine the vertical asymptotes of the curve.

Case II. If the equation of the given curve cannot be put in the form $y = \dfrac{N(x)}{D(x)}$, then we arrange the equation of the curve in the descending powers of y so that it is

$$y^n \phi(x) + y^{n-1} \phi_1(x) + \ldots + \phi_n(x) = 0 \qquad \ldots(1)$$

where $\phi(x)$, $\phi_1(x)$, $\phi_2(x)$, ... are polynomials in x.

Dividing (1) by y^n, we get,

$$\phi(x) + \frac{1}{y}\phi_1(x) + \frac{1}{y^2}\phi_2(x) + \ldots + \frac{1}{y^n}\phi_n(x) = 0 \qquad \ldots(2)$$

We already know, $\underset{y \to \pm\infty}{\text{Lt}} \; x = k$

$\therefore$ proceeding to the limits, (2) gives $\phi(k) = 0$

which shows that k is a root of $\phi(x) = 0$

Let $k_1, k_2, \ldots, k_n$ be the roots of $\phi(x) = 0$, then vertical asymptotes are given by $x = k_1$, $x = k_2$, ..., $x = k_n$

By factor theorem $(x - k_1)$, $(x - k_2)$ ..., $(x - k_n)$ the linear factors of the coefficient of the highest degree term in y, equated to zero gives vertical asymptotes.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

(*b*) Asymptotes parallel to x-axis

As above, for asymptotes parallel to x-axis., if equation of curve can be put in the form $x = \dfrac{N(y)}{D(y)}$ [where $N(y)$ and $D(y)$ have no common factor] then the linear factors of $D(y)$ equated to zero give the asymptotes parallel to x-axis.

In case, the equation cannot be put in the above form, then we arrange it in descending powers of x. Then, the linear factors of the coefficient of the highest power of x equated to zero give the asymptotes parallel to the x-axis.

Note 1. Rule to find asymptotes parallel to x-axis.

Equate to zero the real linear factors in the coefficient of highest power of x in the equation of the given curve.

It should be noted properly that if the coefficient of highest power of x in the equation of the given curve is a constant or has no real linear factor, then the curve has no asymptote parallel to x-axis.

Note 2. Rule to find asymptotes parallel to y-axis.

Equate to zero the real linear factors in the coefficient of highest power of y in the equation of the given curve.

It should be noted properly that if the coefficient of highest power of y in the equation of the given curve is a constant or has no real linear factor, then the curve has no asymptote parallel to y-axis.

ILLUSTRATIVE EXAMPLES

Example 1. Write down, by inspection or otherwise, the vertical and horizontal asymptotes of the curve $3xy + 5x - 4y - 3 = 0$.

Sol. The equation of the given curve is $3xy + 5x - 4y - 3 = 0$ ...(1)

The coefficient of highest power of x in (1) is $3y + 5$

$\therefore$ $3y + 5 = 0$ is the only asymptote parallel to x-axis

$\therefore$ horizontal asymptote of given curve is $3y + 5 = 0$

The coefficient of highest power of y in (1) is $3x - 4$

$\therefore$ $3x - 4 = 0$ is the only asymptote parallel to y-axis

$\therefore$ vertical asymptote of the given curve is $3x - 4 = 0$.

Example 2. Find the asymptotes parallel to the axes of the curve $x^2y^2 + y^2 = 1$.

Sol. The equation of the given curve is $x^2y^2 + y^2 = 1$ ...(1)

The coefficient of highest power of x in (1) is y^2

$\therefore$ $y^2 = 0$ *i.e.*, $y = 0$ is the only asymptote parallel to the x-axis

The coefficient of highest power of y in (1) is $x^2 + 1$. Now $x^2 + 1$ has no real linear factor.

$\therefore$ given curve has no asymptote parallel to y-axis.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education, Bantala, Jaipur

EXERCISE 8 (*a*)

1. Find the asymptotes (if any) parallel to the co-ordinate axes of the following curves :

(*i*) $x^2 y^3 + x^2 + 3y^2 - 9xy + 8x - 25 = 0$

(*ii*) $x^2 y - 3x^2 - 5xy + 6y + 2 = 0$ (*iii*) $2x^3 + x^2 y + y^2 = 4$

(*iv*) $x^2 + y^2 = 4x^2 + 9y^2$ (*v*) $(x+y)^2 = x^2 y^2 - 7$

(*vi*) $y = \dfrac{x}{(x-1)(x-2)}$ (*vii*) $y = x(x-1)(x-2)$

ANSWERS

1. (*i*) $y + 1 = 0$; $x = 0$ (*ii*) $y - 3 = 0$; $x - 2 = 0$, $x - 3 = 0$

(*iii*) No asymptote

(*iv*) No asymptote

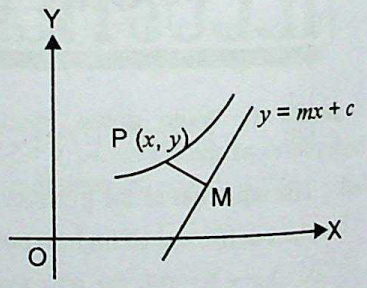
(*v*) $y - 1 = 0$, $y + 1 = 0$; $x - 1 = 0$, $x + 1 = 0$

(*vi*) $y = 0$; $x - 1 = 0$, $x - 2 = 0$ (*vii*) No asymptote

---

Art-4. Oblique Asymptotes

Find the condition that $y = mx + c$ may be asymptote to the curve $y = f(x)$.



Let $y = mx + c$ be an oblique asymptote of the curve $y = f(x)$ where m and c are finite.

Let us determine m and c.

Take a point P (x, y) on the curve

Draw PM $\perp$ on the asymptote

$$y = mx + c \qquad \ldots(1)$$

Then $|PM| = \dfrac{|y - mx - c|}{\sqrt{1 + m^2}}$

As $x \to +\infty$ or $x \to -\infty$, $|PM| \to 0$

$\therefore$ $y - mx - c \to 0$ as $x \to \pm\infty$.

$$\therefore \quad \underset{x \to \pm\infty}{\text{Lt}} (y - mx - c) = 0$$

$$\therefore \quad \underset{x \to \pm\infty}{\text{Lt}} (y - mx) = c$$

Again $\underset{x \to \pm\infty}{\text{Lt}} \left(\dfrac{y}{x} - m\right) = \underset{x \to \pm\infty}{\text{Lt}} \dfrac{y - mx}{x} = \underset{x \to \pm\infty}{\text{Lt}} \dfrac{c}{x} = 0$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

So if $y = mx + c$ is an oblique asymptote of the curve, then

$$\underset{x \to \pm\infty}{\text{Lt}} \frac{y}{x} = m \text{ and } \underset{x \to \pm\infty}{\text{Lt}} (y - mx) = c.$$

Note. Rule to find oblique asymptotes

(*i*) Find $\underset{x \to \infty}{\text{Lt}} \frac{y}{x}$ in the equation of the curve and denote it by m.

(*ii*) Find $\underset{x \to \infty}{\text{Lt}} (y - mx)$ in the equation of the curve and denote it by c.

Then $y = mx + c$ is an asymptote of the curve $f(x, y) = 0$.

Art-5. Asymptote of the General Rational Algebraic Curve

Let the equation of the curve be

$$x^n \phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right) + \ldots + x\phi_1\left(\frac{y}{x}\right) + \phi_0\left(\frac{y}{x}\right) = 0 \quad \ldots(1)$$

where $\phi_n\left(\frac{y}{x}\right)$ represents a polynomial in $\frac{y}{x}$ of degree n.

Dividing (1) by x^n, we get,

$$\phi_n\left(\frac{y}{x}\right) + \frac{1}{x}\phi_{n-1}\left(\frac{y}{x}\right) + \frac{1}{x^2}\phi_{n-2}\left(\frac{y}{x}\right) + \ldots + \frac{1}{x^{n-1}}\phi_1\left(\frac{y}{x}\right) + \frac{1}{x^n}\phi_0\left(\frac{y}{x}\right) = 0$$

Let $x \to \infty$ along the curve, we get,

$$\phi_n(m) = 0 \quad \ldots(2)$$

$$\left[\because \underset{x \to \infty}{\text{Lt}} \frac{y}{x} = m, \text{ (slope of the asymptote)}\right]$$

Roots of (2) determine slopes of various asymptotes of the curve (1).

Let m_1 be one of the roots of (2)

Let $y = m_1 x + c_1$ be the corresponding asymptote.

Suppose $y - m_1 x = p_1$

$\therefore \frac{y}{x} - m_1 = \frac{p_1}{x}$ $\quad \therefore p_1 \to c_1$ as $x \to \infty$

$\therefore \frac{y}{x} = m_1 + \frac{p_1}{x}$ where $p_1 \to c_1$ as $x \to \infty$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Substituting the value of $\frac{y}{x}$ in (1), we get,

$$x^n \phi_n\left(m_1+\frac{p_1}{x}\right)+x^{n-1}\phi_{n-1}\left(m_1+\frac{p_1}{x}\right)+x^{n-2}\phi_{n-2}\left(m_1+\frac{p_1}{x}\right)$$

$$+\ldots+x\phi_1\left(m_1+\frac{p_1}{x}\right)+\phi_0\left(m_1+\frac{p_1}{x}\right)=0 \quad \ldots(3)$$

Expanding each term by Taylor's Theorem, we get,

$$x^n\left[\phi_n(m_1)+\frac{p_1}{x}\phi'_n(m_1)+\frac{p_1^2}{2x^2}\phi''_n(m_1)+\ldots\right]$$

$$+x^{n-1}\left[\phi_{n-1}(m_1)+\frac{p_1}{x}\phi'_{n-1}(m_1)+\frac{p_1^2}{2x^2}\phi''_{n-1}(m_1)+\ldots\right]$$

$$+x^{n-2}\left[\phi_{n-2}(m_1)+\frac{p_1}{x}\phi'_{n-2}(m_1)+\frac{p_1^2}{2x^2}\phi''_{n-2}(m_1)+\ldots\right]+\ldots=0$$

$$\therefore\ x^n\phi_n(m_1)+x^{n-1}\left[p_1\,\phi'_n(m_1)+\phi_{n-1}(m_1)\right]$$

$$+x^{n-2}\left[\frac{p_1^2}{2}\phi''_n(m_1)+p_1\phi'_{n-1}(m_1)+\phi_{n-2}(m_1)\right]+\ldots=0 \quad \ldots(4)$$

Putting $\phi_n(m_1)=0$ and dividing the remaining terms of (4) by x^{n-1} we get,

$$\left[p_1\,\phi'_n(m_1)+\phi_{n-1}(m_1)\right]+\frac{1}{x}\left[\frac{p_1^2}{2}\phi''_n(m_1)+p_1\phi'_{n-1}(m_1)+\phi_{n-2}(m_1)\right]+\ldots=0 \quad \ldots(5)$$

Letting $x\to\infty$, and using $p_1\to c_1$, (5) gives

$$c_1\phi'_n(m_1)+\phi_{n-1}(m_1)=0$$

$$\therefore\quad c_1=-\frac{\phi_{n-1}(m_1)}{\phi'_n(m_1)}\quad(\text{provided } \phi'_n(m_1)\neq 0)$$

Corresponding asymptote is

$$y=m_1x-\frac{\phi_{n-1}(m_1)}{\phi'_n(m_1)}$$

Similarly if $m_2, m_3\ldots$ are the roots of (2), we have corresponding asymptotes

$$y=m_2x-\frac{\phi_{n-1}(m_2)}{\phi'_n(m_2)},\ y=m_3x-\frac{\phi_{n-1}(m_3)}{\phi'_n(m_3)},\ldots$$

provided $\phi'_n(m_2)\neq 0, \phi'_n(m_3)\neq 0, \ldots$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Exceptional Case. When $\phi'_n(m_1) = 0$

Suppose $\phi_{n-1}(m_1) \neq 0$

$c_1 \phi'_n(m_1) + \phi_{n-1}(m_1) = 0$ does not determine any finite value of c_1. Hence there is no asymptote corresponding to the root m_1 of (2).

Now suppose $\phi'_n(m_1) = 0$ and $\phi_{n-1}(m_1) = 0$

In this case the equation $c_1 \phi'_n(m_1) + \phi_{n-1}(m_1) = 0$ becomes an identity.

Letting $x \to \infty$, (4) gives [using $p_1 \to c_1$]

$$\frac{c_1^2}{2} \phi''_n(m_1) + c_1 \phi'_{n-1}(m_1) + \phi_{n-2}(m_1) = 0$$

which is quadratic in c_1 and determines two values c'_1, c''_1 (say), provided $\phi''_n(m_1) \neq 0$.

Then $y = m_1 x + c'_1$ and $y = m_1 x + c''_2$ are the two parallel asymptotes corresponding to the slope m_1.

Note 1. Rule to find oblique asymptotes of a rational algebraic curve :

Step I. Find $\phi_n(m)$, $\phi_{n-1}(m)$ by putting $x = 1$ and $y = m$ in the nth degree terms and in the $(n-1)$th degree terms respectively of the given curve $f(x, y) = 0$.

Step II. Find all the real roots of $\phi_n(m) = 0$.

Step III. If m_1 is a non-repeated root of $\phi_n(m) = 0$, then the corresponding value of c is given by $c\,\phi'_n(m) + \phi_{n-1}(m) = 0$, provided $\phi'_n(m_1) \neq 0$.

If $\phi'_n(m_1) = 0$, then there is no asymptote to the curve corresponding to the value m_1 of m.

Step IV. If m_1 is a repeated root occurring twice, then the corresponding values of c are given by

$$\frac{c^2}{\lfloor 2} \phi''_n(m) + c\,\phi'_{n-1}(m) + \phi_{n-2}(m) = 0, \text{ provided } \phi''_n(m_1) \neq 0.$$

In this case there are two parallel asymptotes to the curve.

Similarly we can proceed when m_1 is repeated three or more times.

Note 2. Rule explained above does not give us vertical asymptotes.

Note 3. A rational algebraic curve of degree n cannot have more than n asymptotes.

ILLUSTRATIVE EXAMPLES

Example 1. Find the asymptotes of the curve $y = \dfrac{x^2 + 2x - 1}{x}$.

Sol. Here equation of curve is $y = \dfrac{x^2 + 2x - 1}{x}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

or $\quad x\,y = x^2 + 2\,x - 1$

or $\quad x^2 - x\,y + 2\,x - 1 = 0 \qquad \ldots(1)$

The coefficient of highest power of x is 1, which is constant.

$\therefore$ given curve has no asymptote parallel to x-axis

The coefficient of highest power of y is $-x$

$\therefore$ the asymptote of given curve parallel to y-axis is $-x = 0$ or $x = 0$

Let us now find oblique asymptote $y = m\,x + c$.

To determine m and c :

$$m = \underset{x \to \infty}{\text{Lt}} \frac{y}{x} = \underset{x \to \infty}{\text{Lt}} \frac{x^2 + 2\,x - 1}{x^2} = \underset{x \to \infty}{\text{Lt}} \left(1 + \frac{2}{x} - \frac{1}{x^2}\right) = 1$$

$$c = \underset{x \to \infty}{\text{Lt}} (y - m\,x) = \underset{x \to \infty}{\text{Lt}} \left(\frac{x^2 + 2\,x - 1}{x} - 1.x\right)$$

$$= \underset{x \to \infty}{\text{Lt}} \frac{x^2 + 2\,x - 1 - x^2}{x} = \underset{x \to \infty}{\text{Lt}} \frac{2\,x - 1}{x} = \underset{x \to \infty}{\text{Lt}} \left(2 - \frac{1}{x}\right) = 2 - 0 = 2.$$

Therefore oblique asymptote is given by $y = x + 2$

Hence the given curve has two asymptotes given by $x = 0$ and $y = x + 2$

Example 2. Find all the asymptotes of the curve

$$x^3 + 2\,x^2\,y - x\,y^2 - 2\,y^3 + 4\,y^2 + 2\,x\,y + y - 1 = 0.$$

Sol. Given equation is

$$x^3 + 2\,x^2\,y - x\,y^2 - 2\,y^3 + 4\,y^2 + 2\,x\,y + y - 1 = 0 \qquad \ldots(1)$$

(1) is an equation of degree 3 in x and y

Since coefficient of x^3 is 1, which is constant

So there is no asymptotes parallel to x-axis

Similarly coefficient of y^3 is -2, which is constant

$\therefore$ there is no asymptote parallel to y-axis.

For oblique asymptotes, put $y = m\,x + c$ in (1), we get,

$$x^3 + 2\,x^2\,(m\,x + c) - x\,(m\,x + c)^2 - 2\,(m\,x + c)^3 + 4\,(m\,x + c)^2 + 2\,x\,(m\,x + c) + (m\,x + c) - 1 = 0.$$

or $\quad x^3\,(1 + 2\,m - m^2 - 2\,m^3) + x^2\,(2\,c - 2\,m\,c - 6\,m^2\,c + 4\,m^2 + 2\,m) + x\,(-c^2 - 6\,m\,c^2 + 8\,m\,c + 2\,c + m) + (-2\,c^3 + 4\,c^2 + c - 1) = 0.$

Equating the coefficient of x^3 and x^2 to zero, we get,

$$1 + 2\,m - m^2 - 2\,m^3 = 0 \qquad \ldots(2)$$

$$2\,c - 2\,m\,c - 6\,m^2\,c + 4\,m^2 + 2\,m = 0 \qquad \ldots(3)$$

From (2), $\quad 1\,(1 + 2\,m) - m^2\,(1 + 2\,m) = 0$

$\therefore \qquad (1 - m^2)\,(1 + 2\,m) = 0$

$\therefore \qquad (1 - m)\,(1 + m)\,(1 + 2\,m) = 0$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

$\therefore \qquad m = 1, -1, -\frac{1}{2}$

When $m = 1$, from (3), we have

$$2c - 2c - 6c + 4 + 2 = 0$$

$\therefore \qquad 6c = 6 \quad$ or $\quad c = 1.$

Corresponding asymptote is $y = x + 1$

When $m = -1$, from (3), we have,

$$2c + 2c - 6c + 4 - 2 = 0$$

$$2c = 2 \quad \text{or} \quad c = 1$$

$\therefore$ corresponding asymptote is $y = -x + 1$

When $m = \frac{-1}{2}$, from (3), we have,

$$2c + c - \frac{3}{2}c + 1 - 1 = 0 \qquad \text{or} \qquad c = 0$$

Corresponding asymptote is $y = -\frac{1}{2}x.$

$\therefore$ given curve has three asymptotes given by $y = x + 1, y = -x + 1$ and $y = -\frac{1}{2}x$.

Example 3. Find the asymptotes of the curve $x^3 + y^3 - 3axy = 0.$

OR

Show that $x + y + a = 0$ is the only asymptote of the curve $x^3 + y^3 - 3axy = 0$

Sol. The equation of given curve is $x^3 + y^3 - 3axy = 0$ ...(1)

The coefficient of highest power of x in (1) is 1, which is constant.

$\therefore$ given curve has no asymptote parallel to x-axis

The coefficient of highest power of y in (1) is 1, which is constant.

$\therefore$ given curve has no asymptote parallel to y-axis.

For oblique asymptotes

Now putting $x = 1$, $y = m$ in (1), we have

$$\phi_3(m) = 1 + m^3 \Rightarrow \phi'_3(m) = 3m^2$$

$$\phi_2(m) = -3am$$

Now $\phi_3(m) = 0$

$\Rightarrow \quad 1 + m^3 = 0 \qquad \Rightarrow (1 + m)(1 - m + m^2) = 0$

$\Rightarrow \quad m = -1, \dfrac{1 \pm \sqrt{1-4}}{2} \qquad \Rightarrow m = -1, \dfrac{1 \pm i\sqrt{3}}{2}$

$\therefore$ only real value of m is -1.

$\therefore \quad m = -1.$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

The corresponding value of c is given by

$$c\,\phi'_3(m) + \phi_2(m) = 0 \qquad [\text{provided } \phi'_3(m) \neq 0]$$

or $c\,[3(-1)^2] + [-3a(-1)] = 0$

$\therefore\ 3c + 3a = 0 \quad \Rightarrow\ c = -a.$

$\therefore$ corresponding asymptote is $y = -x - a$ $\qquad [\because\ y = mx + c]$

or $x + y + a = 0$

$\therefore$ given curve has only one asymptote $x + y + a = 0$.

Example 4. Find the asymptotes of the curve $x^3 + 2x^2y - xy^2 - 2y^3 + xy - y^2 - 1 = 0$.

(J.U. 2016)

Sol. The equation of given curve is $x^3 + 2x^2y - xy^2 - 2y^3 + xy - y^2 - 1 = 0$ ...(1)

The coefficient of highest power of x in (1) is 1, which is constant.

$\therefore$ there is no asymptote parallel to x-axis

The coefficient of highest power of y in (1) is -2, which is constant

$\therefore$ there is no asymptote parallel to y-axis.

For oblique asymptotes

$\phi_3(m) = 1 + 2m - m^2 - 2m^3$ [Putting $x = 1$ and $y = m$ in degree 3 terms]

$\phi_2(m) = m - m^2$

$\phi_1(m) = 0$

$\phi_0(m) = -1$

Now $\phi_3(m) = 0$

$\therefore\ 1 + 2m - m^2 - 2m^3 = 0$

$\Rightarrow 1(1 + 2m) - m^2(1 + 2m) = 0 \quad \Rightarrow\ (1 - m^2)(1 + 2m) = 0$

$\Rightarrow\ (1 - m)(1 + m)(1 + 2m) = 0$

$\therefore\ m = 1, -1, -\dfrac{1}{2}$

$\phi'_3(m) = 2 - 2m - 6m^2$

When $m = 1$, $c = -\dfrac{\phi_2(m)}{\phi'_3(m)} = -\dfrac{\phi_2(1)}{\phi'_3(1)} = -\dfrac{1-1}{2-2-6} = 0$

Corresponding asymptotes is $y = x$, or $x - y = 0$

When $m = -1$, $c = -\dfrac{\phi_2(m)}{\phi'_3(m)} = \dfrac{-1-1}{2+2-6} = \dfrac{-2}{-2} = -1$

Corresponding asymptote is $y = -1x - 1$ *i.e.*, $x + y + 1 = 0$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

When $m = -\frac{1}{2}$, $c = -\frac{\phi_2(m)}{\phi'_3(m)} = -\frac{\phi_2\left(-\frac{1}{2}\right)}{\phi'_3\left(-\frac{1}{2}\right)} = -\frac{-\frac{1}{2}-\frac{1}{4}}{2+1-\frac{3}{2}} = \frac{\frac{3}{4}}{\frac{3}{2}} = \frac{1}{2}$

Corresponding asymptotes is $y = -\frac{1}{2}x + \frac{1}{2}$ *i.e.*, $x + 2y = 1$

Hence asymptotes of the given curve are $x - y = 0$, $x + y + 1 = 0$, $x + 2y = 1$.

Example 5. Find the asymptotes of the curve

$$x^3 - x^2 y - x y^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0.$$

Sol. The equation of given curve is

$$x^3 - x^2 y - x y^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0 \qquad \ldots(1)$$

The coefficient of highest power of x in (1) is 1, which is constant.

$\therefore$ there is no asymptote parallel to x-axis.

The coefficient of highest power of y in (1) is 1, which is constant

$\therefore$ there is no asymptote parallel to y-axis

For oblique asymptotes, we have

$\phi_3(m) = 1 - m - m^2 + m^3 \qquad \therefore\ \phi'_3(m) = -1 - 2m + 3m^2$

$\phi_2(m) = 2 - 4m^2 + 2m \qquad \therefore\ \phi''_3(m) = -2 + 6m$

$\phi_1(m) = 1 + m \qquad \therefore\ \phi'_2(m) = -8m + 2$

$\phi_0(m) = 1$

$\phi_3(m) = 0$ gives

$$1 - m - m^2 + m^3 = 0$$

$$\therefore \quad 1(1-m) - m^2(1-m) = 0$$

$$\Rightarrow \quad (1-m)(1-m^2) = 0$$

$$\Rightarrow \quad (1-m)(1-m)(1+m) = 0$$

$$\therefore \quad m = 1, 1, -1$$

When $m = -1$, $c = -\frac{\phi_2(m)}{\phi'_3(m)} = -\frac{\phi_2(1)}{\phi'_3(1)}$

$$\therefore \quad c = -\frac{2-4-2}{-1+2+3} = 1$$

Corresponding asymptote is $y = -1 \cdot x + 1$ *i.e.*, $x + y = 1$

When $m = 1, 1$, the values of c are given by

$$\frac{c^2}{\lfloor 2} \phi''_3(m) + c\,\phi'_2(m) + \phi_1(m) = 0$$

$$\Rightarrow \quad \frac{c^2}{2}(-2+6) + c(-8+2) + (1+1) = 0$$

$$\Rightarrow \quad 2c^2 - 6c + 2 = 0$$

$$\Rightarrow \quad c^2 - 3c + 1 = 0$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\Rightarrow \quad c = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

and corresponding asymptotes are given by

$$y = 1 \,.\, x + \frac{3+\sqrt{5}}{2}, \quad y = 1x + \frac{3-\sqrt{5}}{2}$$

Hence the required asymptotes are

$$x + y = 1, \; x - y + \frac{3+\sqrt{5}}{2} = 0, \; x - y + \frac{3-\sqrt{5}}{2} = 0.$$

Example 6. Show that the parabola $y^2 = 4\,a\,x$ has no asymptotes.

Sol. The equation of parabola is

$$y^2 = 4\,a\,x \quad \text{or} \quad y^2 - 4\,a\,x = 0 \qquad \ldots(1)$$

The coefficient of highest power of x in (1) is $-4\,a$, which is constant.

$\therefore$ given parabola has no asymptote parallel to x-axis

Also the coefficient of highest power of y in (1) is 1, which is constant.

$\therefore$ given parabola has no asymptote parallel to y-axis.

For oblique asymptotes, we have,

$\phi_2(m) = m^2, \quad \therefore \; \phi'_2(m) = 2\,m$ [Putting $x = 1,\; y = m$ in second degree terms]

$\phi_1(m) = -4\,a$

Now $\phi_2(m) = 0$ given us

$m^2 = 0 \;\Rightarrow\; m = 0, 0.$

For $m = 0$, $\phi'_2(m) = 0$ and $\phi_1(m) = -4\,a \neq 0$

$\therefore$ there is no asymptote corresponding to the value $m = 0$.

$\therefore$ given parabola has no asymptotes.

EXERCISE 8 (*b*)

1. Find all the asymptotes of the following curves :

(*i*) $x^3 + x^2 y + x y^2 + y^3 + 2x^2 + 3xy - 4y^2 + 7x + 2y = 0$

(*ii*) $y^3 - x^2 y - 2xy^2 + 2x^3 - 7xy + 3y^2 + 2x^2 + 2x + 2y + 1 = 0$

(*iii*) $x^3 + 6x^2 y + 11xy^2 + 6y^3 + 3x^2 + 12xy + 11y^2 + 2x + 3y + 5 = 0$

(*iv*) $y^3 - 3x^2 y + xy^2 - 3x^3 + 2y^2 + 2xy + 4x + 5y + 6 = 0$

(*v*) $3x^3 + 2x^2 y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0$

(*vi*) $x^3 - y^3 - x^2 - 5y^2 - 11x - 5y + 7 = 0$

(*vii*) $2x^3 - x^2 y - 2xy^2 + y^3 + 2x^2 + 2y^2 + 2x + 2y + 1 = 0$

(*viii*) $x^4 - y^4 + xy = 0$

(*ix*) $(x + 2y)(x - 2y)(x - y) - 2x(x - 4y) + 2x = 0$

(*x*) $a y^2 = x^2 (a - x)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

2. Find all the asymptotes of the following curves :

(*i*) $x^3 - 5x^2y + 8xy^2 - 4y^3 + x^2 - 3xy + 2y^2 - 7 = 0$

(*ii*) $x^3 - 4x^2y + 5xy^2 - 2y^3 + 3x^2 - 4xy + 2y^2 - 3x + 2y - 1 = 0$

(*iii*) $y^3 + x^2y + 2xy^2 - y + 1 = 0$

(*iv*) $x^3 + 4x^2y + 5xy^2 + 2y^3 + 2x^2 + 4xy + 2y^2 - x - 9y + 1 = 0$

(*v*) $y^3 + 4xy^2 + 4x^2y + 5y^2 + 15xy + 10x^2 - 2x + 1 = 0$

(*vi*) $3x^3 + 17x^2y + 21xy^2 - 9y^3 - 2ax^2 - 12axy - 18ay^2 - 3a^2x + a^2y = 0$

(*vii*) $x^4 - 2x^3y + 2xy^3 - y^4 + x^3 - 4x^2y + 5xy^2 - 2y^3 + x^2 - 3xy + 2y^2 + 1 = 0$

(*viii*) $(x+y)^2(x^2 + xy + y^2) = a^2x^2 + a^3(y - x)$

(*ix*) $(x-y)^2(x-2y)(x-3y) - 2a(x^3 - y^3) - 2a^2(x-2y)(x+y) = 0.$

3. Show that the following curves have no asymptotes :

(*i*) $x^2(y^2 + x^2) = a^2(x^2 - y^2)$ (*ii*) $x^4 + y^4 = a^2(x^2 - y^2)$

4. (*i*) Show that the asymptotes of the curve $x^2y^2 = a^2(x^2 + y^2)$ form a square of side $2a$.

(*ii*) Show that the asymptotes of the curve $(x^2 - y^2)y - 2ay^2 + 5x - 7 = 0$ form a triangle of area a^2.

ANSWERS

1. (*i*) $2x + 2y - 5 = 0$ (*ii*) $x - y - 1 = 0,\ 2x - y = 0,\ x + y + 2 = 0$

(*iii*) $x + y + 1 = 0,\ x + 2y + 1 = 0,\ x + 3y + 1 = 0$

(*iv*) $x + y = 0,\ \sqrt{3}x - y - 1 = 0,\ \sqrt{3}x + y + 1 = 0$

(*v*) $6x - 6y - 7 = 0,\ 6x - 2y - 3 = 0,\ 3x + 6y + 5 = 0$

(*vi*) $x - y - 2 = 0$

(*vii*) $3x + 3y + 2 = 0,\ x - y + 2 = 0,\ 6x - 3y - 10 = 0$

(*viii*) $x - y = 0,\ x + y = 0$

(*ix*) $x + 2y - 2 = 0,\ x - 2y + 2 = 0,\ x - y - 2 = 0$

(*x*) No asymptote

2. (*i*) $x - y = 0,\ x - 2y = 0,\ x - 2y + 1 = 0$

(*ii*) $x - 2y + 6 = 0$ (*iii*) $y = 0,\ x + y - 1 = 0,\ x + y + 1 = 0$

(*iv*) $x + 2y + 2 = 0,\ x + y + 2\sqrt{2} = 0,\ x + y - 2\sqrt{2} = 0$

(*v*) $2y + 5 = 0,\ 2x + y + 2 = 0,\ 4x + 2y + 1 = 0$

(*vi*) $3x - y - 2a = 0,\ x + 3y + a = 0,\ x + 3y - a = 0$

(*vii*) $2x + 2y + 3 = 0,\ x - y = 0,\ 2x - 2y + 1 = 0,\ x - y - 1 = 0$

(*viii*) $x + y + a = 0,\ x + y - a = 0$

(*ix*) $x - y - a = 0,\ x - y - 2a = 0,\ x - 2y + 14a = 0,\ x - 3y - 13a = 0$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Art-6. Special Methods for Finding Oblique Asymptotes of a Rational Algebraic Curve

Now we discuss some special methods of finding asymptotes of $f(x, y) = 0$ when the equation $f(x, y) = 0$ is of some special types.

Method I. If the equation of the curve is of the form

$$(ax + by + c)\, f_{n-1}(x, y) + g_{n-1}(x, y) = 0$$

then the asymptote parallel to $ax + by + c = 0$ is given by

$$(ax + by + c) + \underset{\substack{x \to \infty \\ \frac{y}{x} \to -\frac{a}{b}}}{\text{Lt}} \frac{g_{n-1}(x, y)}{f_{n-1}(x, y)} = 0, \text{ provided the limit exists}$$

Method II. If the equation of the curve is of the form

$$(ax + by)^2\, f_{n-2}(x, y) + g_{n-2}(x, y) = 0$$

then the two asymptotes parallel to $ax + by = 0$ are given by

$$(ax + by)^2 + \underset{\substack{x \to \infty \\ \frac{y}{x} \to -\frac{a}{b}}}{\text{Lt}} \frac{g_{n-2}(x, y)}{f_{n-2}(x, y)} = 0, \text{ provided the limit exists}$$

Method III. If the equation of the curve is of the form

$$(ax + by)^2 f_{n-2}(x, y) + (ax + by)\, g_{n-2}(x, y) + h_{n-2}(x, y) = 0$$

then the two asymptotes parallel to $ax + by = 0$ are given by

$$(ax + by)^2 + (ax + by) \underset{\substack{x \to \infty \\ \frac{y}{x} \to -\frac{a}{b}}}{\text{Lt}} \frac{g_{n-2}(x, y)}{f_{n-2}(x, y)} + \underset{\substack{x \to \infty \\ \frac{y}{x} \to -\frac{a}{b}}}{\text{Lt}} \frac{h_{n-2}(x, y)}{f_{n-2}(x, y)} = 0$$

provided the limits exist.

Note. Working Method

(*i*) Factorize the highest degree terms

(*ii*) Retain one linear factor and divide by the product of other factors.

(*iii*) Take limits when $x \to \infty$, $y \to \infty$ in the direction of the retained factor.

Note. If limits do not exist, then there is no asymptote parallel to $ax + by + c = 0$.

Art-7. Asymptotes by Inspection

If the equation of the curve can be written as

$$F_n(x, y) + F_{n-2}(x, y) = 0.$$

where $F_n(x, y)$ is a rational integral function in x and y of degree n and $F_{n-2}(x, y)$ of degree $(n - 2)$ at the most then every linear factor $ax + by + c$ of $F_n(x, y)$ equated to zero determines the asymptote of the curve, provided no two asymptotes so obtained are either parallel or coincident.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

ILLUSTRATIVE EXAMPLES

Example 1. Find all the asymptotes of the curve

$$x^3 - 2y^3 + xy(2x - y) + y(x - y) + 1 = 0.$$

Sol. The equation of the curve is

$$x^3 - 2y^3 + xy(2x - y) + y(x - y) + 1 = 0 \qquad \ldots(1)$$

or $$x^3 + 2x^2y - xy^2 - 2y^3 + y(x - y) + 1 = 0$$

or $$(x - y)(x + y)(x + 2y) + y(x - y) + 1 = 0 \qquad \ldots(2)$$

[$\because$ of factorizing third degree terms]

$\therefore$ possible asymptotes are parallel to the lines

$x - y = 0,\ x + y = 0$ and $x + 2y = 0.$

(*i*) The equation (2) of the curve can be written as

$$x - y + \frac{y(x - y) + 1}{(x + y)(x + 2y)} = 0$$

$\therefore$ asymptote (if it exists) parallel to $x - y = 0$ is given by

$$x - y + \underset{\substack{x \to \infty \\ y = x}}{\text{Lt}} \frac{y(x - y) + 1}{(x + y)(x + 2y)} = 0$$

or $$x - y + \underset{x \to \infty}{\text{Lt}} \frac{0 + 1}{(x + x)(x + 2x)} = 0 \quad \text{or} \quad x - y + \underset{x \to \infty}{\text{Lt}} \frac{1}{6x^2} = 0$$

or $x - y = 0,$ which is one asymptote.

(*ii*) The equation (2) of the curve can be written as

$$x + y + \frac{y(x - y) + 1}{(x - y)(x + 2y)} = 0$$

$\therefore$ asymptote (if it exists) parallel to $x + y = 0$ is given by

$$x + y + \underset{\substack{x \to \infty \\ y = -x}}{\text{Lt}} \frac{y(x - y) + 1}{(x - y)(x + 2y)} = 0$$

or $$x + y + \underset{x \to \infty}{\text{Lt}} \frac{(-x)(2x) + 1}{(2x)(-x)} = 0$$

or $$x + y + \underset{x \to \infty}{\text{Lt}} \frac{1 - 2x^2}{-2x^2} = 0 \quad \text{or} \quad x + y + \underset{x \to \infty}{\text{Lt}} \frac{\frac{1}{x^2} - 2}{-2} = 0$$

or $$x + y + \frac{0 - 2}{-2} = 0$$

or $x + y + 1 = 0,$ which is second asymptote.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

(*iii*) The equation (2) of the curve can be written as

$$x+2y+\frac{y(x-y)+1}{(x-y)(x+y)}=0$$

$\therefore$ asymptote (if any) parallel to $x+2y=0$ is given by

$$x+2y+\underset{\substack{x\to\infty\\ y=-\frac{x}{2}}}{\text{Lt}}\frac{y(x-y)+1}{(x-y)(x+y)}=0$$

or $$x+2y+\underset{x\to\infty}{\text{Lt}}\frac{\left(-\frac{x}{2}\right)\left(x+\frac{x}{2}\right)+1}{\left(x+\frac{x}{2}\right)\left(x-\frac{x}{2}\right)}=0$$

or $$x+2y+\underset{x\to\infty}{\text{Lt}}\frac{-\frac{3x^2}{4}+1}{\frac{3x^2}{4}}=0 \quad \text{or} \quad x+2y+\underset{x\to\infty}{\text{Lt}}\frac{4-3x^2}{3x^2}=0$$

or $$x+2y+\underset{x\to\infty}{\text{Lt}}\frac{\frac{4}{x^2}-3}{3}=0 \quad \text{or} \quad x+2y+\frac{-3}{3}=0$$

or $x+2y-1=0$, which is third asymptote.

$\therefore$ asymptotes of the given curve are $x-y=0$, $x+y+1=0$ and $x+2y-1=0$.

Example 2. Find all the asymptotes of the following curve :

$$x^3+x^2y-xy^2-y^3+2xy+2y^2-3x+y=0$$

Sol. The given equation is $x^3+x^2y-xy^2-y^3+2xy+2y^2-3x+y=0$

or $x^2(x+y)-y^2(x+y)+2xy+2y^2-3x+y=0$

or $(x+y)(x^2-y^2)+2xy+2y^2-3x+y=0$

or $(x-y)(x+y)^2+2xy+2y^2-3x+y=0$ ...(1)

The equation (1) can be written as

$$x-y+\frac{2xy+2y^2-3x+y}{(x+y)^2}=0$$

$\therefore$ asymptote (if it exists) parallel to $x-y=0$ is given by

$$x-y+\underset{\substack{x\to\infty\\ y=x}}{\text{Lt}}\frac{2xy+2y^2-3x+y}{(x+y)^2}=0$$

or $$x-y+\underset{x\to\infty}{\text{Lt}}\frac{2x^2+2x^2-3x+x}{(x+x)^2}=0$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

or $\quad x - y + \underset{x \to \infty}{\text{Lt}} \dfrac{4x^2 - 2x}{4x^2} = 0$

or $\quad x - y + \underset{x \to \infty}{\text{Lt}} \dfrac{4 - \dfrac{2}{x}}{4} = 0 \qquad$ or $\quad x - y + \dfrac{4-0}{4} = 0$

$\therefore \quad x - y + 1 = 0$ is one asymptote.

The equation (1) can be written as

$$(x+y)^2 + (x+y) \cdot \frac{2y}{x-y} - \frac{3x-y}{x-y} = 0$$

$\therefore$ asymptotes (if they exist) parallel to $x + y = 0$ are given by

$$(x+y)^2 + (x+y) \cdot \underset{\substack{x \to \infty \\ y=-x}}{\text{Lt}} \frac{2y}{x-y} - \underset{\substack{x \to \infty \\ y=-x}}{\text{Lt}} \frac{3x-y}{x-y} = 0$$

or $\quad (x+y)^2 + (x+y) \cdot \underset{x \to \infty}{\text{Lt}} \dfrac{-2x}{x+x} - \underset{x \to \infty}{\text{Lt}} \dfrac{3x+x}{x+x} = 0$

or $\quad (x+y)^2 - (x+y) - 2 = 0 \qquad$ or $\quad (x+y-2)(x+y+1) = 0$

$\therefore \quad x + y - 2 = 0, \; x + y + 1 = 0$ are the other two asymptotes.

Example 3. Find all the asymptotes of the following curve :

$$xy(x^2 - y^2)(x^2 - 4y^2) + 3xy(x^2 - y^2) + x^2 + y^2 - 7 = 0$$

Sol. The equation of given curve is

$$xy(x^2 - y^2)(x^2 - 4y^2) + 3xy(x^2 - y^2) + x^2 + y^2 - 7 = 0$$

The given equation is of the form

$$F_6(x, y) + F_4(x, y) = 0$$

where $F_6(x, y) = xy(x^2 - y^2)(x^2 - 4y^2)$, $F_4(x, y) = 3xy(x^2 - y^2) + x^2 + y^2 - 7$

Also $F_6(x, y)$ is the product of 6 linear factors (non-repeated) and $F_4(x, y)$ is of degree 4.

$\therefore$ asymptotes are given by $F_6(x, y) = 0$

or $\quad xy(x^2 - y^2)(x^2 - 4y^2) = 0 \quad$ or $\quad xy(x-y)(x+y)(x-2y)(x+2y) = 0$

$\therefore$ asymptotes of the given curve are

$$x = 0, \; y = 0, \; x - y = 0, \; x + y = 0, \; x - 2y = 0, \; x + 2y = 0.$$

EXERCISE 8 (c)

1. Find all asymptotes of the curve

$$f(x, y) = x(y - x)^3 - x(y - x) + 2 = 0$$

2. Find asymptotes of the curve $x^2 y - xy^2 + xy + y^2 + x - y = 0$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College of Education Bantalab Jammu

3. Find all the asymptotes of the following curves

(*i*) $(x+y-1)(x-y+2)(2x+y)+3x^2+2xy+y^2+4x=0$

(*ii*) $(x+y)(x+2y)(x+3y)+3x^2+12xy+11y^2+x+y+2=0$

(*iii*) $(x-1)(x-2)(x+y)+x^2+x+1=0$

(*iv*) $xy(x^2-y^2)+2x^2+2xy+2y^2-1=0$

(*v*) $(x^2-y^2)(x+2y+1)+x+y+1=0.$

4. Find all the asymptotes of the following curves :

(*i*) $(x-y+1)(x-y-2)(x+y)=8x-1$

(*ii*) $(x+2y-1)(x^2+y^2)+2x^2-y^2+4x-5=0$

(*iii*) $(x+y+1)(x^2-xy+y^2)-6xy+2x^2+y^2+3x-7y+4=0$

(*iv*) $(x+y)^2(x^2+y^2)-8(x+y)y^2+4x^2-3xy-y^2+4x+3=0$

(*v*) $(y-x)(y-2x)^2+(y+3x)(y-2x)+2x+2y-1=0$

(*vi*) $(x^2-y^2)^2+(x^3+y^3)+xy+7=0$

ANSWERS

1. $x=0,\ y-x=0,\ y-x-1=0,\ y-x+1=0$

2. $y=0,\ x=1,\ y=x+2$

3. (*i*) $x+y=0,\ x-y+3=0,\ 2x+y-1=0$

(*ii*) $x+y+1=0,\ x+2y+1=0,\ x+3y+1=0$

(*iii*) $x-1=0,\ x-2=0,\ x+y+1=0$

(*iv*) $x=0,\ y=0,\ y+x=0,\ y-x=0$

(*v*) $x-y=0,\ x+y=0,\ x+2y+1=0$

4. (*i*) $x+y=0,\ x-y-3=0,\ x-y+2=0$

(*ii*) $5x+10y+2=0$

(*iii*) $x+y+4=0$

(*iv*) $x+y-1=0,\ x+y-3=0$

(*v*) $x-y+4=0,\ 2x-y-2=0,\ 2x-y-3=0$

(*vi*) $x+y+1=0,\ 4x+4y-1=0$

Art-8. Intersection of a Curve and its Asymptotes

Prove that an asymptote of a rational algebraic curve of the nth degree cuts the curve in atmost $(n-2)$ points.

Proof. Let $y=m_1x+c_1$ ...(1) be an asymptote of the curve

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$x^n \phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right) + \ldots = 0 \qquad \ldots(2)$$

We are to find the points of intersection of (1) and (2),

From (1), $\frac{y}{x} = m_1 + \frac{c_1}{x}$

Substituting the value of $\frac{y}{x}$ in (2), we get,

$$x^n \phi_n\left(m_1 + \frac{c_1}{x}\right) + x^{n-1}\phi_{n-1}\left(m_1 + \frac{c_1}{x}\right) + x^{n-2}\phi_{n-2}\left(m_1 + \frac{c_1}{x}\right) + \ldots = 0$$

Using Taylor's Theorem, we get

$$x^n \phi_n(m_1) + x^{n-1}\left[c_1 \phi'_n(m_1) + \phi_{n-1}(m_1)\right]$$

$$+ x^{n-2}\left[\frac{c_1^2}{2}\phi''_n(m_1) + c_1\phi'_{n-1}(m_1) + \phi_{n-2}(m_1)\right] + \ldots = 0 \qquad \ldots(3)$$

Since $\phi_n(m_1) = 0$ and $c_1\phi'_n(m_1) + \phi_{n-1}(m_1) = 0$, (3) becomes

$$x^{n-2}\left[\frac{c_1^2}{2}\phi''_n(m_1) + c_1\phi'_{n-1}(m_1) + \phi_{n-2}(m_1)\right] + \ldots = 0$$

which is an equation of degree $(n-2)$ and correspondingly (1) and (2) intersect in $(n-2)$ points.

$\therefore$ asymptote (1) cuts the curve (2) in at the most $(n-2)$ points.

Hence the result.

Cor. 1. Prove that all asymptotes of a curve of nth degree cut the curve in atmost $n(n-2)$ points.

Proof. We know that a curve of nth degree has atmost n asymptotes and each asymptote cuts the curve in atmost $(n-2)$ points.

$\therefore$ all the asymptotes of a curve of nth degree cut the curve in atmost $n(n-2)$ points.

Cor. 2. If the equation of the curve of nth degree is of the form $F_n + F_{n-2} = 0$ and curve has no parallel asymptotes, then the points of intersection of the curve and its asymptote lie on the curve $F_{n-2} = 0$.

Proof. The equation of curve is $F_n + F_{n-2} = 0$

The equation of asymptote is $F_n = 0$

$\therefore$ the points of intersection of the asymptote and the curve satisfy the equations $F_n + F_{n-2} = 0$ and $F_n = 0$ and therefore they will satisfy

$$(F_n + F_{n-2}) - F_n = 0 \quad i.e., \quad F_{n-2} = 0.$$

Hence the result.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

ILLUSTRATIVE EXAMPLES

Example 1. Show that the asymptotes of the cubic curve $x^3 - xy^2 - 2xy + 2x - y - 1 = 0$ cut the curve in atmost three points which lies on the line $3x - y - 1 = 0$.

Sol. The equation of given curve is $x^3 - xy^2 - 2xy + 2x - y - 1 = 0$

Coefficient of highest power of y is $-x$.

$\therefore$ asymptote parallel to y-axis is $-x = 0$ or $x = 0$

The equation (1) can be written as $x(x-y)(x+y) - 2xy + 2x - y - 1 = 0$

The asymptote (if it exists) parallel to $x - y = 0$ is given by

$$x - y + \underset{\substack{x \to \infty \\ y = x}}{\text{Lt}} \frac{-2xy + 2x - y - 1}{x(x+y)} = 0$$

or $$x - y + \underset{x \to \infty}{\text{Lt}} \frac{-2x^2 + 2x - x - 1}{x(x+x)} = 0$$

or $$x - y + \underset{x \to \infty}{\text{Lt}} \frac{-2x^2 + x - 1}{2x^2} = 0 \quad \text{or} \quad x - y + \underset{x \to \infty}{\text{Lt}} \frac{-2 + \frac{1}{x} - \frac{1}{x^2}}{2} = 0$$

or $$x - y + \frac{-2 + 0 - 0}{2} = 0$$

$\therefore$ $x - y - 1 = 0$ is the second asymptote.

The asymptote (if it exists) parallel to $x + y = 0$ is given by

$$x + y + \underset{\substack{x \to \infty \\ y = -x}}{\text{Lt}} \frac{-2xy + 2x - y - 1}{x(x-y)} = 0$$

or $$x + y + \underset{x \to \infty}{\text{Lt}} \frac{2x^2 + 2x + x - 1}{x(x+x)} = 0$$

or $$x + y + \underset{x \to \infty}{\text{Lt}} \frac{2x^2 + 3x - 1}{2x^2} = 0 \quad \text{or} \quad x + y + \underset{x \to \infty}{\text{Lt}} \frac{2 + \frac{3}{x} - \frac{1}{x^2}}{2} = 0$$

or $$x + y + \frac{2 + 0 - 0}{2} = 0$$

$\therefore$ $x + y + 1 = 0$ is the third asymptote.

The joint equation of the asymptotes is

$$x(x - y - 1)(x + y + 1) = 0 \quad \text{or} \quad x^3 - xy^2 - 2xy - x = 0 \qquad \ldots(2)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

 Gandhi Memorial College Of Education Bantalab Jammu

Subtracting (2) from (1), we get,

$3x - y - 1 = 0$, which is a straight line

Now curve (1) is of third degree and has 3 asymptotes. Therefore the curve and the asymptotes intersect in atmost 3 (3 – 2) = 3 points which lie on the straight line $3x - y - 1 = 0$.

Example 2. Show that the asymptotes of the curve

$$x^4 - 5x^2y^2 + 4y^4 + x^2 - y^2 + x + y + 1 = 0$$

cut the curve in atmost eight points which lie on a rectangular hyperbola.

Sol. The equation of given curve is

$$x^4 - 5x^2y^2 + 4y^4 + x^2 - y^2 + x + y + 1 = 0 \qquad ...(1)$$

Coefficient of highest power of x is 1, which is constant

$\therefore$ there is no asymptote parallel to x-axis

Similarly there is no asymptote parallel to y-axis

Putting $x = 1, y = m$ in (1), we get

$$\phi_4(m) = 1 - 5m^2 + 4m^4 = 4m^4 - 5m^2 + 1$$

$$\phi_4'(m) = 16m^3 - 10m$$

$$\phi_3(m) = 0$$

Now $\phi_4(m) = 0 \Rightarrow 4m^4 - 5m^2 + 1 = 0 \Rightarrow (m^2 - 1)(4m^2 - 1) = 0$

$\therefore \quad m^2 = 1, \frac{1}{4} \Rightarrow m = -1, 1, -\frac{1}{2}, \frac{1}{2}$

When $m = -1$, $c = -\dfrac{0}{16(-1)^3 - 10(-1)} = 0$ $\qquad \left[\because c = -\dfrac{\phi_3(m)}{\phi_4'(m)}\right]$

$\therefore$ corresponding asymptote is $y = -x + 0$ or $x + y = 0$

When $m = 1$, $c = -\dfrac{0}{16(1)^3 - 10(1)} = 0$

$\therefore$ corresponding asymptote is $y = x + 0$ or $x - y = 0$

When $m = -\frac{1}{2}$, $c = -\dfrac{0}{16\left(-\frac{1}{2}\right)^3 - 10\left(-\frac{1}{2}\right)} = 0$

$\therefore$ corresponding asymptote is $y = -\frac{1}{2}x + 0$ or $x + 2y = 0$

When $m = \frac{1}{2}$, $c = -\dfrac{0}{16\left(\frac{1}{2}\right)^3 - 10\left(\frac{1}{2}\right)} = 0$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$\therefore$ corresponding asymptote is $y = \frac{1}{2}x + 0$ or $x - 2y = 0$

The joint equation of asymptotes is

$$(x-y)(x+y)(x-2y)(x+2y) = 0$$

or $(x^2 - y^2)(x^2 - 4y^2) = 0$

or $x^4 - 5x^2y^2 + 4y^4 = 0$ ...(2)

Subtracting (2) from (1), we get

$x^2 - y^2 + x + y + 1 = 0$, which is a rectangular hyperbola.

Now curve (1) is of 4th degree and has 4 asymptotes. Therefore the curve and the asymptotes intersect in atmost 4 (4 – 2) = 8 points which lie on the rectangular hyperbola $x^2 - y^2 + x + y + 1 = 0$.

Example 3. Find the equation of the cubic which has the same asymptotes as the curve $x^3 - 6x^2y + 11xy^2 - 6y^3 + 4x + 5y + 7 = 0$ and which passes through the points (0, 0), (– 2, 0) and (0, – 2).

Sol. The equation of given curve is $x^3 - 6x^2y + 11xy^2 - 6y^3 + 4x + 5y + 7 = 0$ ...(1)

It is of the form $F_3 + F_1 = 0$

$\therefore$ asymptotes are given by $F_3 = 0$

or $x^3 - 6x^2y + 11xy^2 - 6y^3 = 0$ or $(x-y)(x-2y)(x-3y) = 0$

$\therefore$ asymptotes of (1) are $x - y = 0$, $x - 2y = 0$, $x - 3y = 0$

The equation of the cubic curve which has the same asymptotes is of the type

$(x-y)(x-2y)(x-3y) + ax + by + c = 0$ ...(2)

Now (2) passes through (0, 0), $\therefore$ $c = 0$

(2) passes through (– 2, 0), $\therefore$ $-8 - 2a = 0 \Rightarrow a = -4$

(2) passes through (0, – 2), $\therefore$ $48 - 2b = 0 \Rightarrow b = 24$

Substituting values of a, b, c in (2), we get,

$$(x-y)(x-2y)(x-3y) - 4x + 24y = 0$$

or $x^3 - 6x^2y + 11xy^2 - 6y^3 - 4x + 24y = 0.$

EXERCISE 8 (*d*)

1. Show that the asymptotes of the curve

$$(x^2 - y^2)(y^2 - 4x^2) + 6x^3 - 5x^2y - 3xy^2 + 2y^3 - x^2 + 3xy - 1 = 0$$

cut the curve in atmost eight points which lie on a unit circle.

2. Find the asymptotes of the curve

$$3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0.$$

Show that the asymptotes meet the curve again at three points which lie on a line. Find its equation.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

 Gandhi Memorial College Of Education Bantalab Jammu

3. Find the asymptotes of the curve $x^2 y + x y^2 + 2 x^2 - 2 x y - y^2 - 6 x - 2 y + 2 = 0$ and show that they cut the curve in atmost three points which lie on the straight line $2 x - 3 y - 4 = 0$.

4. Show that the asymptotes of the curve
$$x^3 - 2 y^3 + x y (2 x - y) + y (x - y) + 1 = 0$$
meet the curve in three points which lie on the line $x - y + 1 = 0$.

5. Show that the three asymptotes of the curve
$$x^3 + 2 x^2 y - x y^2 - 2 y^3 + 4 y^2 + 2 x y + y - 1 = 0$$
meet the curve again in three points which lie on the line $x + 3 y = 1$.

6. Show that the asymptotes of the curve
$$(2 x - 3 y + 1)^2 (x + y) - 8 x + 2 y - 9 = 0$$
meet the curve in three points which lie on the line $4 x - 6 y + 9 = 0$.

7. Show that the four asymptotes of the curve
$$4 x^4 - 13 x^2 y^2 + 9 y^4 + 32 x^2 y - 42 y^3 - 20 x^2 + 74 y^2 - 56 y + 4 x + 16 = 0$$
meet the curve in eight points which lie on the curve $y^2 = -4 x$.

8. Show that the points of intersection of the curve $x y (x^2 - y^2) - 25 x^2 - 9 y^2 + 144 = 0$ and its asymptotes lie on ellipse whose eccentricity is $\frac{4}{5}$.

9. Find the equation of the cubic which has the same asymptotes as the curve $x^2 y - x y^2 + x y + y^2 + x - y = 0$ and passes through the points (0, 0), (0, 1) and (1, 0).

10. Find the equation of the cubic curve which has the same asymptotes as the curve $x^3 - 6 x^2 y + 11 x y^2 - 6 y^3 + x + y + 1 = 0$ and which passes through the points (0, 0), (2, 0) and (0, 2).

11. Find the equation of the hyperbola having $x + y + 1 = 0$ and $x - y - 2 = 0$ as its asymptotes and passing through the origin.

12. Find the equation of the curve which has $y = 0$, $x = 0$, $x \pm y = 0$ as asymptotes and these asymptotes cut the curve again in eight points lying on the circle $x^2 + y^2 = a^2$.

ANSWERS

2. $381 x - 106 y - 105 = 0$

9. $x^2 y - x y^2 + x y + y^2 - y = 0$

10. $x^3 - 6 x^2 y + 11 x y^2 - 6 y^3 - 4 x + 24 y = 0$

11. $(x + y + 1)(x - y - 2) + 2 = 0$

12. $b x y (x^2 - y^2) + a (b^2 - a^2)(x^2 + y^2 - a^2) = 0$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

9

DOUBLE POINTS

Art-1. Singular Point

A point on the curve at which the curve behaves in an extraordinary manner is called a singular point.

There are two types of singular points :

(*i*) Points of inflexion (*ii*) Multiple points

We have already discussed points of inflexion. Now we discuss multiple points.

Multiple Point. A point on the curve through which more than one branches of the curve pass is called a multiple point.

In this chapter, we are mainly concerned with double points. Now we define double point.

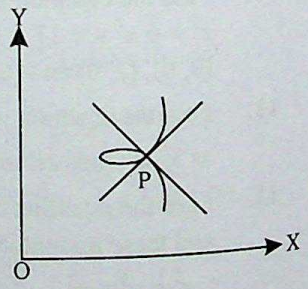
Double Point. A point on the curve through which two branches of the curve pass is called a double point.

Classification of Double Points

There are three kinds of double points.

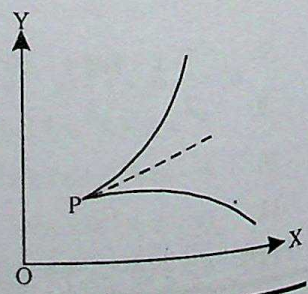
(*i*) Node :

A node is a point on the curve through which two real branches of the curve pass and two tangents at which are real and distinct. Thus P is a node.



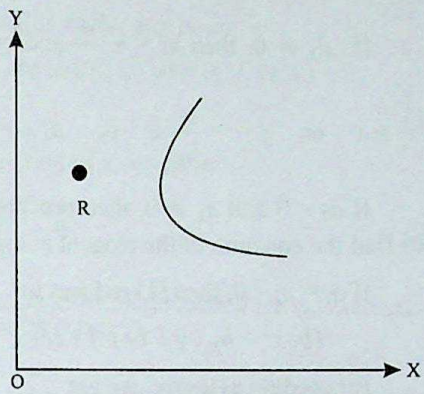
(*ii*) Cusp :

A double point on the curve through which two real branches of the curve pass and the tangents at which are real and coincident is called a cusp. Thus P is a cusp.



CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

(*iii*) **Conjugate Point :** A conjugate point on a curve is a point in the neighbourhood of which there are no other real points of the curve.



The two tangents at a conjugate point are in general imaginary but sometimes they may be real.

Note 1. Conjugate point is also called *isolated point.*

Note 2. The determination of the nature of double points depends basically on the nature of the two branches of the curve passing through it and not on the tangents to the curve at that point. Generally when the tangents at a double point are real, the branches are also real. But there are cases, when the tangents may be real, yet the branches may be imaginary.

Art- 2. Tangents at the Origin

If a rational algebraic curve $f(x, y) = 0$ passes through the origin, then the equation (or equations) of the tangent (or tangents) at the origin is (or are) obtained by equating to zero the lowest degree terms of $f(x, y)$.

Proof : Let the equation of curve be

$$f(x, y) = (a_1 x + a_2 y) + (b_1 x^2 + b_2 xy + b_3 y^2)$$
$$+ (c_1 x^3 + c_2 x^2 y + c_3 xy^2 + c_4 y^3) + \ldots\ldots \quad \ldots(1)$$

[$\because$ the curve passes through origin, so its equation does not contain any constant term.]

Let P (α, β) be any point in the nbd. of O (0, 0) on the curve (1).

$\therefore$ slope of chord OP $= \dfrac{\beta - 0}{\alpha - 0} = \dfrac{\beta}{\alpha}$

Now chord OP becomes a tangent at O as P $\to$ O i.e. $\alpha \to 0, \beta \to 0$

$\therefore$ $m = \underset{P \to O}{\text{Lt}} \dfrac{\beta}{\alpha}$, where m is the slope of the tangent at O.

$\because$ P (α, β) lies on curve (1)

$\therefore$ $(a_1 \alpha + a_2 \beta) + (b_1 \alpha^2 + b_2 \alpha\beta + b_3 \beta^2) + \ldots = 0$

Dividing throughout by α $(\neq 0)$, we get

$$\left(a_1 + a_2 \frac{\beta}{\alpha}\right) + \left(b_1 \alpha + b_2 \beta + b_3 \beta . \frac{\beta}{\alpha}\right) + \ldots = 0 \quad \ldots(2)$$

Let P $\to$ O so that $\alpha \to 0, \beta \to 0$ and $\underset{P \to O}{\text{Lt}} \dfrac{\beta}{\alpha} = m$

$\therefore$ from (2), we get

$a_1 + a_2 m = 0$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

If $a_2 \neq 0$, then $m = -\dfrac{a_1}{a_2}$ and hence the equation of the tangent at the origin is

$$y = m\,x \quad \text{or} \quad y = -\frac{a_1}{a_2}\,x \quad \text{or} \quad a_1 x + a_2 y = 0$$

If $a_2 = 0$ and $a_1 \neq 0$, then proceeding as above but considering slope w.r.t. y-axis, we find the equation of the tangent at origin is $a_1 x + a_2 y = 0$

If $a_1 = a_2 = 0$, then (1) reduces to

$$(b_1 x^2 + b_2 x y + b_3 y^2) + (c_1 x^3 + c_2 x^2 y + c_3 x y^2 + c_4 y^3) + \ldots = 0 \qquad \ldots(3)$$

Proceeding as above, we get

$$b_1 + b_2 m + b_3 m^2 = 0$$

Putting $\dfrac{y}{x}$ for m, the equation of the two tangents at origin is

$$b_1 x^2 + b_2 x y + b_3 y^2 = 0$$

If $b_1 = b_2 = b_3 = 0$, we proceed as above and so on.

$\therefore$ it follows that *equation of tangent (or tangents) is obtained by equating to zero the lowest degree terms in* (x, y).

Note : Equation of tangents at any point

Shift the origin to the given point and find the equation of tangents at the new origin. Then transform this equation to original axes.

Art-3. Working rule for finding the Nature of Origin which is a Double Point

Find the tangents at the origin by equating to zero the lowest degree terms in x and y of the equation of the curve. If the origin is a double point, then we shall get two tangents which may be real or imaginary.

(*i*) If two tangents are imaginary, then origin is a conjugate point.

(*ii*) If two tangents real and coincident, then origin is a cusp or a conjugate point.

(*iii*) If the two tangents are real and distinct, then origin is a node or a conjugate point.

To be sure, examine the nature of curve in the nbd. of origin. If the curve has real branches through the origin, then it is a node, otherwise a conjugate point.

To be sure, we test the nature of curve in the nbd. of the origin as above.

Note. Test for nature of curve at origin.

If the tangents at origin are $y^2 = 0$, solve the equation of the curve for y, neglecting all terms of y containing powers above second. If the values of y, for small values of x are found to be real, the branches of the curve through the origin are real, otherwise imaginary.

If the tangents at origin are $x^2 = 0$, solve the equation for x and proceed as above.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Art-4. Show that the necessary and sufficient conditions for any point (x, y) on $f(x, y) = 0$ to be a multiple point are that $f_x(x, y) = 0$, $f_y(x, y) = 0$.

Proof. The equation of curve is $f(x, y) = 0$ ...(1)

Differentiating (1) w.r.t. x, treating y as a function of x, we have

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0 \qquad ...(2)$$

where $\frac{dy}{dx}$ is the slope of the tangent at the point P (x, y)

If P (x, y) is a multiple point, there must be atleast two tangents which may be real, coincident or imaginary.

Thus $\frac{dy}{dx}$ must have atleast two values at (x, y). But (2) is a first degree equation in $\frac{dy}{dx}$ and is satisfied by atleast two values of $\frac{dy}{dx}$, which is possible only when it is an identity.

$$\therefore \quad \frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0 \qquad ...(3)$$

$\therefore$ necessary and sufficient conditions for any point (x, y) on the curve $f(x, y) = 0$ to be a multiple point is

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

i.e. $f_x(x, y) = 0$ and $f_y(x, y) = 0$

Classification of Double Points

Differentiating (2) w.r.t. x, we get

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y} \cdot \frac{dy}{dx} + \left(\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial y^2} \frac{dy}{dx} \right) \frac{dy}{dx} + \frac{\partial f}{\partial y} \cdot \frac{d^2 y}{dx^2} = 0$$

But $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ and at a double point $\frac{\partial f}{\partial y} = 0$

$\therefore$ above equation becomes

$$\frac{\partial^2 f}{\partial y^2} \left(\frac{dy}{dx} \right)^2 + 2 \frac{\partial^2 f}{\partial x \partial y} \frac{dy}{dx} + \frac{\partial^2 f}{\partial x^2} = 0 \qquad ...(4)$$

This is a quadratic in $\frac{dy}{dx}$ and gives the two slopes of the tangents at the double point (x, y).

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

The tangents at (x, y) will be real and distinct; real and coincident or imaginary according as the roots of (4) are real and different, real and equal or imaginary for which

$$4\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 - 4\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} \gtreqless 0 \qquad \left[\because \text{disc}, = b^2 - 4ac \gtreqless 0\right]$$

or
$$\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\,\frac{\partial^2 f}{\partial y^2} \gtreqless 0$$

Hence, in general, a double point will be a node, a cusp or a conjugate point according as

$$\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 \gtreqless \frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} \qquad ...(5)$$

at that point.

Note. *The condition (5) is not a sure test for the node, cusp or a conjugate point. This in fact is the condition for the two tangents at the double points to be real and distinct, coincident or imaginary. This result can lead us to wrong conclusion.* We explain it by an example.

Example : Find the position and nature of the double points on the curve

$$a^4 y^2 = x^4 (2x^2 - 3a^2).$$

Sol. The equation of given curve is

$$f(x, y) = 2x^6 - 3a^2x^4 - a^4y^2 = 0 \qquad ...(1)$$

$$\therefore \qquad \frac{\partial f}{\partial x} = 12x^5 - 12a^2x^3, \quad \frac{\partial f}{\partial y} = -2a^4 y$$

Now for the double points, $\dfrac{\partial f}{\partial x} = 0, \dfrac{\partial f}{\partial y} = 0$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 12x^5 - 12a^2x^3 = 0 \Rightarrow x^3(x^2 - a^2) = 0$$

$\Rightarrow \quad x = 0,\ a,\ -a$

and $\dfrac{\partial f}{\partial y} = 0 \qquad \Rightarrow \quad -2a^4 y = 0 \quad \Rightarrow y = 0$

$\therefore$ the possible double points are $(0, 0), (a, 0), (-a, 0)$

But $(a, 0)$ $(-a, 0)$ do not satisfy (1)

$\therefore$ $(0, 0)$ is the only double point.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Now $\dfrac{\partial^2 f}{\partial x^2} = 60x^4 - 36a^2x^2$, $\dfrac{\partial^2 f}{\partial y^2} = -2a^4$, $\dfrac{\partial^2 f}{\partial x \partial y} = 0$

At (0, 0), $\dfrac{\partial^2 f}{\partial x^2} = 0$, $\dfrac{\partial^2 f}{\partial y^2} = -2a^4$, $\dfrac{\partial^2 f}{\partial x \partial y} = 0$

$$\therefore \quad \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} = (0)^2(-2a)^4 = 0$$

$\therefore$ origin will be a cusp

But from (1), $y = \pm \dfrac{x^2}{a^2}\sqrt{2x^2 - 3a^2}$

For small enough values of x (+ ve or − ve), $2x^2 - 3a^2$ is − ve and so y is imaginary. Hence no portion of the curve lies in the neighbourhood of the origin.

$\therefore$ *origin will be conjugate point*

$\therefore$ above method gives a wrong conclusion. So for greater accuracy, we must proceed as explained in working method given below.

Art-5. Working rule for finding the Position and Nature of Double Points of the Curve $f(x, y) = 0$

Step I. Find $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^2 f}{\partial y^2}$

Step II. Solve the equations $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$ to get possible double points.

Reject those points which do not satisfy the equation $f(x, y) = 0$ of the curve.

Remaining are the double points

Step III. At each double point, calculate $D = \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2 - \dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2}$

(*a*) If D is positive, double point is a node or conjugate point

(*b*) If D = 0, double point is a cusp or conjugate point.

In these cases (*a*) and (*b*), find the nature by shifting the origin to the double points and then testing the nature of tangents and existence of the curve in the nbd. of new origin.

(*c*) If D is negative, double point is a conjugate point.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

ILLUSTRATIVE EXAMPLES

Example 1. Examine the nature of origin for the curves :

(*i*) $x^4 - a x^2 y + a x y^2 + a^2 y^2 = 0,\ a > 0$ (*ii*) $x^4 + y^4 - 4 a x y = 0$

(*iii*) $y^3 = x^3 + a x^2,\ a > 0$

Sol. (*i*) The equation of curve is $x^4 - a x^2 y + a x y^2 + a^2 y^2 = 0$ ...(1)

Equating to zero, the lowest degree terms, the tangents at the origin are given by

$a^2 y^2 = 0$ *i.e.* $y^2 = 0$ *i.e.* $y = 0,\ y = 0$

$\therefore$ there are two real and coincident tangents at the origin

$\therefore$ *origin is either a cusp or a conjugate point*

To be sure, we study the nature of given curve near the origin.

From (1), $a(x + a)\ y^2 - a x^2 y + x^4 = 0$

$$\therefore \quad y = \frac{a x^2 \pm \sqrt{a^2 x^4 - 4 a x^4 (x+a)}}{2 a (x+a)} \quad \Rightarrow \quad y = \frac{a x^2 \pm x^2 \sqrt{-4 a x - 3 a^2}}{2 a (x+a)}$$

Now, for small values of $x \neq 0$, $-4 a x - 3 a^2$ is negative and so y is imaginary in the nbd. of origin.

$\therefore$ origin is a conjugate point

Note. Here (0, 0) is a conjugate point and tangents at (0, 0) are real.

$\therefore$ tangents at a conjugate point can be real

(*ii*) The equation of curve is $x^4 + y^4 - 4 a x y = 0$ ...(1)

Equating to zero, the lowest degree terms, the tangents at the origin are given by

$-4 a x y = 0$ or $x y = 0$ or $x = 0, y = 0$

$\because$ the tangents are real and distinct

$\therefore$ origin is a node or a conjugate point

From (1), neglecting y^4, we get

$$x^4 - 4 a x y = 0 \qquad \text{or} \quad y = \frac{1}{4a} x^3$$

$\therefore$ y is real for values of x near origin

$\therefore$ origin is a node

(*iii*) The equation of curve is $y^3 = x^3 + a x^2$ ...(1)

Equating to zero, the lowest degree terms, the tangents at the origin are given by

$a x^2 = 0$ or $x^2 = 0$ or $x = 0, 0$

$\because$ the two tangents are real and coincident

$\therefore$ origin is a cusp or a conjugate point

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

From (1), neglecting x^3, we get

$$a x^2 = y^3 \quad \text{or} \quad x = \pm y \sqrt{\frac{y}{a}}$$

Since $a > 0$, therefore x is real for small positive values of y

$\therefore$ the two branches of the curve near the origin are real and so the origin is a cusp.

Example 2. Show that origin is a node, a cusp or a conjugate point on the curve

$y^2 = a x^2 + b x^3$, according as a is positive, zero or negative.

Sol. The equation of curve is $y^2 = a x^2 + b x^3$ ...(1)

Equating to zero, the lowest degree terms in (1), the tangents at the origin are given by

$$y^2 = a x^2 \quad \text{or} \quad y = \pm \sqrt{a x} \qquad ...(2)$$

Since there are two tangents at the origin

$\therefore$ origin is a double point

Case I. When a is positive

$\therefore$ from (2), the tangents are real and distinct.

$\therefore$ origin is a node or a conjugate point

From (1), $\quad y = \pm x\sqrt{a + b x}$

For small positive values of x, y is real

$\therefore$ there are two real branches of the curve near the origin

$\therefore$ origin is a node.

Case II. When $a = 0$

From (2), tangents are $y = 0$, $y = 0$

$\therefore$ two tangents are real and coincident

$\Rightarrow$ origin is a cusp or a conjugate point

When $a = 0$, from (1),

$y^2 = b x^3$ or $y = \pm x\sqrt{b x}$, which gives real values of x when b and x have like signs

$\therefore$ y is real for values of x near the origin

$\therefore$ origin is a cusp

Case III. When a is negative

From (2), two tangents at origin are imaginary

$\therefore$ origin is a conjugate point.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bahal Jatu

Example 3. Show that the curve $y^2 = bx \sin \frac{x}{a}$ has a node or a conjugate point at the origin according as a and b have like or unlike signs.

Sol. The equation of given curve is $y^2 = bx \sin \frac{x}{a}$

$$\text{or} \quad y^2 = bx\left[\frac{x}{a} - \frac{1}{\lfloor 3}\frac{x^3}{a^3} + \frac{1}{\lfloor 5}\frac{x^5}{a^5} - \ldots\right] \qquad \left[\because \sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \ldots\right]$$

$$\text{or} \quad y^2 = \frac{b}{a}\left[x^2 - \frac{x^4}{6a^2} + \frac{x^6}{120\,a^4} - \ldots\right] \qquad \ldots(1)$$

Equating to zero, the lowest degree terms, the tangents at the origin are given by

$$y^2 = \frac{b}{a}x^2 \quad \text{or} \quad y = \pm x\sqrt{\frac{b}{a}}$$

$\therefore$ there are two distinct tangents at the origin

$\therefore$ origin is a double point

Case I. When a and b have like signs, then $\frac{b}{a}$ is positive and so $\sqrt{\frac{b}{a}}$ is real.

$\therefore$ two tangents are real and distinct

$\therefore$ origin is either a node or a conjugate point

For small values of $x \neq 0$, the behaviour of R.H.S. of (1) is the same as that of its first term *i.e.* (1) behaves as $y^2 = \frac{b}{a}x^2$. Since $\frac{b}{a}$ is positive, so y^2 is positive for small values of x whether positive or negative.

$\therefore$ y is real in the nbd. of origin

$\therefore$ origin is a node.

Case II. When a and b have unlike signs, then $\frac{b}{a}$ is negative and so $\sqrt{\frac{b}{a}}$ is imaginary.

$\therefore$ two tangents are imaginary

$\therefore$ origin is a conjugate point.

Example 4. Prove that the curve $y^2 = (x-a)^2(x-b)$ has at $x = a$, a node if $a > b$, a cusp if $a = b$ and a conjugate point if $a < b$.

Sol. The equation of curve is $y^2 = (x-a)^2(x-b)$ ...(1)

When $x = a$, from (1), $y = 0$

$\therefore$ point under discussion in $(a, 0)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh:Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Shifting origin to $(a, 0)$ by transformations, $x = X + a$, $y = Y + 0 = Y$

(1) becomes $\quad Y^2 = X^2 (X + a - b)$ ...(2)

Equating to zero, the lowest degree terms, the tangents at the new origin are given by

$$Y^2 = X^2 (a-b) \qquad \text{or} \quad Y = \pm X\sqrt{a-b} \qquad ...(3)$$

Case I. When $a > b$

From (3), two tangents at new origin are real and different

$\therefore$ new origin $(a, 0)$ is a node or a conjugate point

From (2), $\quad Y = \pm X \sqrt{X + a - b}$

For small non-zero values of X, Y is real as $a - b > 0$

$\therefore$ new origin $(a, 0)$ is a node

Case II. When $a = b$

From (3), tangents are $Y = 0$, $Y = 0$

$\therefore$ two tangents are real and coincident

$\therefore$ origin is a cusp or a conjugate point

From (2), $Y^2 = X^3 \quad$ or $\quad Y = \pm X \sqrt{X}$

For small positive values of X, Y is real

$\therefore$ new origin $(a, 0)$ is a cusp.

Case III. When $a < b$

From (2), two tangents at new origin are imaginary

$\therefore$ $(a, 0)$ is a conjugate point.

Example 5. Determine the position and nature of the double point on the curve

$$x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0.$$

Sol. The equation of curve is

$$f(x, y) = x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0 \qquad ...(1)$$

$$\therefore \qquad \frac{\partial f}{\partial x} = 3x^2 - 14x + 15, \ \frac{\partial f}{\partial y} = -2y + 4$$

For the double points $\dfrac{\partial f}{\partial x} = 0, \ \dfrac{\partial f}{\partial y} = 0, \ f(x, y) = 0$

Now $\dfrac{\partial f}{\partial x} = 0 \quad \Rightarrow \quad 3x^2 - 14x + 15 = 0$

$\Rightarrow \quad (x - 3)(3x - 5) = 0 \ \Rightarrow \ x = 3, \dfrac{5}{3}$

and $\dfrac{\partial f}{\partial y} = 0 \quad \Rightarrow \quad -2y + 4 = 0 \ \Rightarrow \ y = 2$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College of Education Bantalab Jammu

$\therefore$ the possible double points are $(3, 2), \left(\frac{5}{3}, 2\right)$

But $\left(\frac{5}{3}, 2\right)$ does not satisfy (1)

$\therefore$ (3, 2) is the only double point

Nature of the point (3, 2)

Shifting the origin to the point (3, 2) by transformations $x = X + 3,\ y = Y + 2$

(1) becomes $(X + 3)^3 - (Y + 2)^2 - 7(X + 3)^2 + 4(Y + 2) + 15(X + 3) - 13 = 0$

or $X^3 + 9X^2 + 27X + 27 - Y^2 - 4Y - 4 - 7X^2 - 42X - 63 + 4Y$

$$+ 8 + 15X + 45 - 13 = 0$$

or $X^3 + 2X^2 - Y^2 = 0$ ...(2)

Equating to zero, the lowest degree terms, the tangents at the new origin are given by

$$2X^2 - Y^2 = 0 \quad \text{or} \quad Y = \pm\sqrt{2}\,X$$

which are real and distinct

$\therefore$ new origin is either a node or a conjugate point

From (2), $Y = \pm X\sqrt{X+2}$

which gives real values of Y for small values of X, positive or negative

$\therefore$ real branches of the curve exist in the nbd. of the new origin (3, 2)

$\therefore$ (3, 2) is a node.

Alter. The equation of the curve is

$$f(x, y) = x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0$$

$$\therefore \quad \frac{\partial f}{\partial x} = 3x^2 - 14x + 15,\ \frac{\partial f}{\partial y} = -2y + 4 \qquad ...(1)$$

For the double points $\frac{\partial f}{\partial x} = 0,\ \frac{\partial f}{\partial y} = 0,\ f(x, y) = 0$

Now $\frac{\partial f}{\partial y} = 0 \Rightarrow 3x^2 - 14x + 15 = 0$

$\Rightarrow (x - 3)(3x - 5) = 0 \Rightarrow x = 3, \frac{5}{3}$

and $\frac{\partial f}{\partial y} = 0 \Rightarrow -2y + 4 = 0 \Rightarrow y = 2$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$\therefore$ the possible double points are (3, 2), $\left(\frac{5}{3}, 2\right)$

But $\left(\frac{5}{3}, 2\right)$ does not satisfy (1)

$\therefore$ (3, 2) is the only double point

Nature of the point (3, 2)

$$\frac{\partial^2 f}{\partial x^2} = 6x - 14, \quad \frac{\partial^2 f}{\partial y^2} = -2, \quad \frac{\partial^2 f}{\partial x\, \partial y} = 0$$

At (3, 2)

$$\frac{\partial^2 f}{\partial x^2} = 18 - 14 = 4, \quad \frac{\partial^2 f}{\partial y^2} = -2, \quad \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\therefore \quad \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} = (0)^2 - (4)(-2) = 8 > 0$$

$\therefore$ (3, 2) is node.

Example 6. Find the position and nature of the double points on the curve

$$(x-2)^2 = y\,(y-1)^2. \qquad \text{(J.U. 2016)}$$

Sol. The equation of curve is $f(x, y) = (x-2)^2 - y\,(y-1)^2 = 0$ ...(1)

$$\frac{\partial f}{\partial x} = 2(x-2),$$

$$\frac{\partial f}{\partial y} = -2y\,(y-1) - (y-1)^2 = -(y-1)(3y-1)$$

For the double points $\frac{\partial f}{\partial x} = 0, \ \frac{\partial f}{\partial y} = 0, \ f(x, y) = 0$

Now $\frac{\partial f}{\partial x} = 0 \quad \Rightarrow 2\,(x-2) = 0 \quad \Rightarrow x = 2$

and $\frac{\partial f}{\partial y} = 0 \quad \Rightarrow -(y-1)\,(3y-1) = 0 \quad \Rightarrow y = 1, \frac{1}{3}$

$\therefore$ the possible double points are (2, 1) $\left(2, \frac{1}{3}\right)$

Now $\left(2, \frac{1}{3}\right)$ does not satisfy (1)

$\therefore$ (2, 1) is the only double point

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Nature of the point (2, 1)

$$\frac{\partial^2 f}{\partial x^2} = 2,\ \frac{\partial^2 f}{\partial y^2} = -3(y-1) - (3y-1),\ \frac{\partial^2 f}{\partial x\,\partial y} = 0$$

$$\frac{\partial^2 f}{\partial x^2} = 2,\ \frac{\partial^2 f}{\partial y^2} = -3\,(1-1) - (3-1) = -2,\ \frac{\partial^2 f}{\partial x\,\partial y} = 0$$

$$\therefore\ \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} = (0)^2 - (2)\,(-2) = 4 > 0$$

$\therefore$ there is a node at the point (2, 1).

Example 7. Examine the curve $x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$ for a double point and show that it is a cusp.

Sol. The equation of curve is

$$f(x, y) = x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0 \qquad ...(1)$$

$$\frac{\partial f}{\partial x} = 3x^2 + 4x + 2y + 5$$

$$\frac{\partial f}{\partial y} = 2x - 2y - 2$$

For double points $\frac{\partial f}{\partial x} = 0,\ \frac{\partial f}{\partial y} = 0,\ f(x, y) = 0$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 + 4x + 2y + 5 = 0 \qquad ...(2)$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow x - y - 1 = 0 \Rightarrow x = y + 1 \qquad ...(3)$$

$\therefore$ from (2), $3(y+1)^2 + 4(y+1) + 2y + 5 = 0$

$\therefore$ $3y^2 + 6y + 3 + 4y + 4 + 2y + 5 = 0$

$\therefore$ $3y^2 + 12y + 12 = 0 \Rightarrow y^2 + 4y + 4 = 0 \Rightarrow (y+2)^2 = 0 \Rightarrow y = -2$

$\therefore$ from (3), $x = -2 + 1 = -1$

$\therefore$ possible double point is $(-1, -2)$

It satisfies (1), Therefore $(-1, -2)$ is the only double point

Now $\frac{\partial^2 f}{\partial x^2} = 6x + 4,\ \frac{\partial^2 f}{\partial y^2} = -2,\ \frac{\partial^2 f}{\partial x\,\partial y} = 2$

At (−1, −2)

$$\frac{\partial^2 f}{\partial x^2} = -6 + 4 = -2,\ \frac{\partial^2 f}{\partial y^2} = -2,\ \frac{\partial^2 f}{\partial x\,\partial y} = 2$$

$$D = \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} = (2)^2 - (-2)\,(-2) = 4 - 4 = 0$$

$\therefore$ $(-1, -2)$ is a cusp.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

 Gandhi Memorial College Of Education Bantalab Jammu

Note : Another method of finding the position of double points.

Let (h, k) be the double point of the curve $f(x, y) = 0$. Shift the origin to (h, k) by the transformations $x = X + h$, $y = Y + k$ and let $F(X, Y) = 0$ the transformed equation of curve.

Since the new origin is a double point

$\therefore$ constant term and terms of the first degree in X, Y is $F(X, Y) = 0$ must be absent. Therefore equating to zero the constant term, coefficients of X and Y separately to zero, we get three equations in h, k. Now any value of h, k obtained from solving two equations and satisfying the third is a double point.

Example 8. Find the position and nature of double points on the curve

$$y^2 = (x-1)(x-2)^2.$$

Sol. The equation of curve is $y^2 = (x-1)(x-2)^2$ ...(1)

Let (h, k) be a double point on this curve

Shifting the origin to (h, k) by the transformations $x = X + h$, $y = Y + k$

(1) becomes

$$(Y + k)^2 = (X + h - 1)(X + h - 2)^2$$

or $$Y^2 + 2kY + k^2 = X^3 + 2(h-2)Y^2 + (h-2)^2 X + (h-1)X^2 + 2(h-1)(h-2)X + (h-1)(h-2)^2$$

or $$k^2 - (h-1)(h-2)^2 + 2kY - [(h-2)^2 + 2(h-1)(h-2)]X - (h-1)X^2 - 2(h-2)X^2 + Y^2 - X^3 = 0 \quad ...(2)$$

Since new origin is a double point

$\therefore$ constant term and coefficients of X and Y are separately zero.

$\therefore$ $k^2 - (h-1)(h-2)^2 = 0$ ...(3)

$2k = 0 \quad \Rightarrow k = 0$ ...(4)

and $(h-2)^2 + 2(h-1)(h-2) = 0 \quad \Rightarrow (h-2)[(h-2) + 2(h-1)] = 0$

$\Rightarrow (h-2)(3h+4) = 0 \quad \Rightarrow h = 2, \frac{4}{3}$

Now $h = \frac{4}{3}$, $k = 0$ do not satisfy (3)

$\therefore$ $h = 2, k = 0$ *i.e.* $(2, 0)$ is the only double point

Putting $h = 2$, $k = 0$ in (2), we get,

$$Y^2 = X^3 + X^2 \quad ...(5)$$

Equating to zero, the lowest degree terms, the tangents at new origin are given by

$$Y^2 = X^2 \quad \text{or} \quad Y = \pm X$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$\therefore$ two tangents are real and distinct

$\therefore$ new origin (2, 0) is a node or a conjugate point.

From (5), $Y = \pm X\sqrt{X+1}$

which gives real values of Y for small values of X, positive or negative.

$\therefore$ real branches of the curve exist in the nbd. of the new origin (2, 0)

$\therefore$ (2, 0) is a node.

EXERCISE 9 (a)

1. Examine the nature of origin for the curves :

   (*i*) $y^2 = 2x^2y + x^4y - 2x^4$ (*ii*) $y^2(a^2 + x^2) = x^2(a^2 - x^2)$

   (*iii*) $x^2(x - y) + y^2 = 0$

2. Show that the curve $y^2 = bx \tan \frac{x}{a}$ has a node or a conjugate point at the origin, according as a and b have like or unlike signs.

3. Show that the curve $y^2 = 2x \sin 2x$ has a node at the origin.

4. Find the position and nature of the double points of the following curves :

   (*i*) $x^3 + x^2 + y^2 - x - 4y + 3 = 0$

   (*ii*) $x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0$

   (*iii*) $x^4 - 4ax^3 + 2ay^3 + 4a^2x^2 - 3a^2y^2 - a^4 = 0$

   (*iv*) $x^4 - 4y^3 - 12y^2 - 8x^2 + 16 = 0$

   (*v*) $ay^2 - x^3 + bx^2 = 0$

   (*vi*) $a^2y^2 - b^2x^2 - x^2y^2 = 0$

5. Prove that the only singular point on the curve $(y - b)^2 = (x - a)^3$ is a cusp and find its co-ordinates.

6. Find the position and nature of double points on the following curves :

   (*i*) $2(x^3 + y^3) - 3(3x^2 + y^2) + 12x - 4 = 0$

   (*ii*) $x(x^2 + y^2) = ay^2$

   (*iii*) $x^3 + y^3 = 3axy$

7. Find the position and nature of the double points on the following curves :

   (*i*) $x^4 + y^3 + 2x^2 + 3y^2 = 0$

   (*ii*) $x^4 + 4ax^3 + 4a^2x^2 - b^2y^2 - 2b^3y - a^4 - b^4 = 0$

8. (*i*) Show that at the point (1, −1), there is a cusp on the curve

   $x^3 + xy^2 + y^3 - 4x^2 + y^2 + 4x + y - 1 = 0$

   (*ii*) Show that (0, 0) is a conjugate point on the curve $(x - y)^2 + x^4 = 0$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

 Gandhi Memorial College Of Education Bantalab Jammu

ANSWERS

1. (*i*) Conjugate point (*ii*) node (*iii*) cusp.

4. (*i*) $(-1, 2)$ is a node (*ii*) $(0, -a), (-a, 0), (a, 0)$ are nodes

(*iii*) $(a, 0)$ is a conjugate point (*iv*) $(0, -2), (2, 0)\ (-2, 0)$ are nodes.

(*v*) $(0, 0)$ is a conjugate point if a, b are of same sign, cusp if one of a, b is zero and node if a, b are of opposite signs

(*vi*) $(0, 0)$ is a node

6. (*i*) $(1, 1), (2, 0)$ are nodes (*ii*) $(0, 0)$ is a cusp.

(*iii*) $(0, 0)$ is a node

7. (*i*) $(0, 0)$ is a conjugate point (*ii*) $(-a, -b)$ is a conjugate point

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

10

TRACING OF CURVES IN CARTESIAN COORDINATES

Art-1. Introduction

We have done curve tracing in lower classes by giving different values to x and finding the corresponding values of y. Now we will do curve tracing using idea of asymptotes, monotonocity, maxima and minima, etc.

We also need idea of multiple points and in particular double points. Tangents at the origin are also needed. Therefore, before actually going to curve tracing, we discuss multiple points and tangents at the origin.

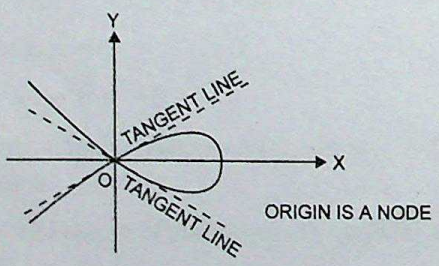
Art-2. Multiple Points

A point, through which two or more than two branches of a curve pass, is called a multiple point.

In particular, if only two branches of a curve pass through a point, the point is called a **double point.**

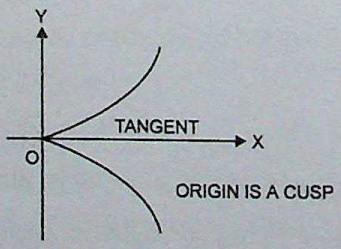
Classification of double points

1. Node : A double point is called a node if the two tangents at the double point are real and distinct (See the figure).



Node, sometimes, is called a node point.

2. Cusp : A double point is called a cusp if the two tangents at the double point are real and coincident. (See the figure)



Cusp, sometimes, is also called a cuspidal point.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

3. Conjugate or Isolated point : A double point is called a conjugate (or an isolated) point if the two tangents at the double point are not real or there is no real point on the curve in the neighbourhood of the double point.

Art-3. Tangents at the Origin

If a rational algebraic curve $f(x, y) = 0$ passes through the origin, then the equation (or equations) of the tangent (or tangents) at the origin is (or are) obtained by equating to zero the lowest degree terms of $f(x, y)$.

Proof : Let the equation of curve be

$$f(x, y) = (a_1 x + a_2 y) + (b_1 x^2 + b_2 x y + b_3 y^2) + (c_1 x^3 + c_2 x^2 y + c_3 x y^2 + c_4 y^3) + \ldots \quad \ldots(1)$$

[$\because$ the curve passes through origin, so its equation does not contain any constant term]

Let P (α, β) be any point in the *nhd.* of O(0, 0) on the curve (1)

$\therefore$ slope of chord OP $= \dfrac{\beta - 0}{\alpha - 0} = \dfrac{\beta}{\alpha}$.

Now chord OP becomes a tangent at

O as P $\to$ O *i.e.*, $\alpha \to 0,\ \beta \to 0$

$\therefore \quad m = \underset{P \to O}{\text{Lt}} \dfrac{\beta}{\alpha}$, where m is the slope of the tangent at O.

$\because$ P (α, β) lie on the curve (1)

$\therefore \quad (a_1 \alpha + a_2 \beta) + (b_1 \alpha^2 + b_2 \alpha \beta + b_3 \beta^2) + \ldots = 0$

Dividing throughout by $\alpha\ (\neq 0)$, we get,

$$\left(a_1 + a_2 \frac{\beta}{\alpha}\right) + \left(b_1 \alpha + b_2 \beta + b_3 \beta . \frac{\beta}{\alpha}\right) + \ldots = 0 \quad \ldots(2)$$

Let P $\to$ O so that $\alpha \to 0,\ \beta \to 0$ and $\underset{P \to O}{\text{Lt}} \dfrac{\beta}{\alpha} = m$

$\therefore$ from (2), we get, $a_1 + a_2 m = 0$

If $a_2 \neq 0$, then $m = -\dfrac{a_1}{a_2}$ and hence the equation of the tangent at the origin is

$y = m x$

or $\quad y = -\dfrac{a_1}{a_2} x \quad$ or $\quad a_1 x + a_2 y = 0.$

If $a_2 = 0$ and $a_1 \neq 0$, then proceeding as above but considering slope w.r.t. y-axis, we find the equation of the tangent at origin is $a_1 x + a_2 y = 0$.

If $a_1 = a_2 = 0$, then (1) reduces to

$$(b_1 x^2 + b_2 x y + b_3 y^2) + (c_1 x^3 + c_2 x^2 y + c_3 x y^2 + c_4 y^3) + \ldots = 0 \quad \ldots(3)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Proceeding as above, we get,

$$b_1 + b_2 m + b_3 m^2 = 0$$

Putting $\frac{y}{x}$ for m, the equation of the two tangents at origin is

$$b_1 x^2 + b_2 x y + b_3 y^2 = 0$$

If $b_1 = b_2 = b_3 = 0$, we proceed as above, and so on.

$\therefore$ it follows that equation of tangent (or tangents) is obtained by equating to zero the lowest degree terms in $f(x, y)$

Note. Equation of tangents at any point

Shift the origin to the given point and find the equation of tangents at the new origin. Then transform this equation to original axes.

Art-4. Curve Tracing

We shall keep in mind the following points for tracing the graph of the equation $f(x, y) = 0$.

I. Symmetry

Curve given by $f(x, y) = 0$ is symmetric about

(*i*) x-axis if it is unchanged on changing y to $-y$ *i.e.*, if $f(x, -y) = f(x, y)$

(*ii*) y-axis if it is unchanged on changing x to $-x$ *i.e.*, if $f(-x, y) = f(x, y)$

(*iii*) the origin if it is unchanged on changing x to $-x$ and y to $-y$

i.e., if $f(-x, -y) = f(x, y)$

(*iv*) the line $y = x$ if it is unchanged on changing x to y and y to x

i.e., if $f(x, y) = f(y, x)$

(*v*) the line $y = -x$ if it is unchanged on changing x to $-y$ and y to $-x$

i.e., if $f(-y, -x) = f(x, y)$.

II. Domain and Range : Find the domain and range.

III. Origin : Check whether origin lies on the curve. If curve passes through origin, then find the tangents at the origin and also determine whether origin is node, cusp or an isolated point.

IV. Asymptotes : Find all the asymptotes of the curve and the position of the curve relative to its asymptotes.

V. Points of Intersection : Find the points of intersection of the curve with co-ordinate axes and obtain the equations of the tangents at these points. If any of these is a double point, then find the nature of the double point.

Also find some other points on the curve by giving suitable values to x.

VI. Maxima and Minima : Find the points where the function has maximum value or minimum value. Also find the maximum and minimum value at each point.

VII. Points of inflexion : (*a*) Find the intervals of

(*i*) increase and decrease of the curve

(*ii*) concavity and convexity of the curve.

(*b*) . Also find the points of inflexion, if any.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

VIII. Discontinuities : Find the points at which function is discontinuous. Also discuss the behavior of the function near these points.

One thing should be kept in mind that with all the information collected as stated above, one may not be in a position to trace the curve. Then there is also the limitation of time in the examination. What is the use of doing one part in 40 or more minutes when only 15 minutes are allotted for one part. It is only intuition, experience and practice that will enable a student to trace a curve properly. A student may not need all the information if he has got sufficient practice to trace the curves. So students should have as much practice as they can have.

We will give some examples to give students some experience of curve tracing.

ILLUSTRATIVE EXAMPLES

Example 1. Trace the curve $y = x^3$.

Sol. The equation of the curve is $y = x^3$ ...(1)

(*i*) **Symmetry :** The curve is symmetrical about the origin as the equation of the curve does not change on changing x to $-x$ and y to $-y$. The curve is symmetrical about the origin means the curve is symmetrical in opposite quadrants.

(*ii*) **Origin :** The curve passes through the origin

The tangents at the origin are $y = 0$ *i.e.*, x-axis

(*iii*) **Domain :** Here $D_f = (-\infty, \infty)$

Now $x > 0 \Rightarrow y > 0$ and $x < 0 \Rightarrow y < 0$

$\therefore$ curve lies either in the first quadrant or in the third quadrant.

(*iv*) **Points of Intersection :** The curve meets x-axis where $y = 0$. Putting $y = 0$ in (1), we get $x = 0$. Therefore curve meets x-axis in (0, 0). Also the curve meets y-axis in (0, 0).

(*v*) **Asymptotes :** The curve has no asymptotes.

(*vi*) **Increasing and decreasing :** $\dfrac{dy}{dx} = 3x^2$

$$\frac{d^2y}{dx^2} = 6x, \quad \frac{d^3y}{dx^3} = 6$$

Now $\dfrac{dy}{dx} = 0 \quad \Rightarrow 3x^2 = 0 \quad \Rightarrow x = 0$

$\therefore$ tangent at $x = 0$ is parallel to x-axis

Now $\dfrac{dy}{dx} = 3x^2 > 0 \ \forall$ real x

$\therefore$ curve is increasing for all real x.

$$\frac{d^2y}{dx^2} = 6x > 0 \text{ when } x > 0$$

$\therefore$ curve is concave upwards in $(0, \infty)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Similarly curve is concave downwards in $(-\infty, 0)$

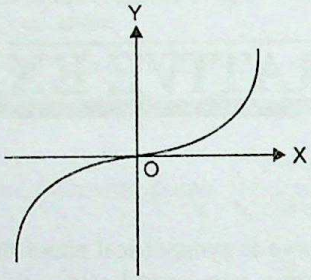
Again $\dfrac{d^2y}{dx^2} = 0$ at $(0, 0)$, but $\dfrac{d^3y}{dx^3} = 6 \neq 0$ at $(0, 0)$

$\therefore$ $(0, 0)$ is a point of inflexion.

(*vii*) **Additional Points**

Now $y \to \infty$ as $x \to \infty$ and also $y \to -\infty$ as $x \to -\infty$

A rough sketch of the curve is shown in the figure.



Example 2. Trace the curve $y = x^3 + 5x^2 + 3x - 4$.

Sol. The equation of curve is $y = x^3 + 5x^2 + 3x - 4$ ...(1)

(*i*) **Symmetry :** The curve is neither symmetrical about axes nor about origin. It is also not symmetrical about $y = x$ or $y = -x$.

(*ii*) **Origin :** The curve does not pass through the origin.

(*iii*) **Domain :** $D_f = (-\infty, \infty)$

(*iv*) **Points of Intersection :** The curve meets x-axis where $y = 0$.

Putting $y = 0$ in (1), we get,

$$x^3 + 5x^2 + 3x - 4 = 0 \qquad ...(2)$$

$x = -4$ satisfies it

$\therefore$ other roots of (2) are given by

$$x^2 + x - 1 = 0$$

$$\therefore \quad x = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2} = \frac{-1 \pm 2{\cdot}2}{2} = 0{\cdot}6, -1{\cdot}6$$

$\therefore$ curve meets x-axis in $(-4, 0)$, $(-1{\cdot}6, 0)$ and $(0{\cdot}6, 0)$

Also the curve meets y-axis where $x = 0$

Putting $x = 0$ in (1), we get, $y = -4$

$\therefore$ curve (1) meets y-axis in $(0, -4)$

(*v*) **Asymptotes :** The curve has no asymptotes.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

(*vi*) **Increasing and decreasing :** Now $\frac{dy}{dx} = 3x^2 + 10x + 3$ and $\frac{d^2y}{dx^2} = 6x + 10$

Now $\frac{dy}{dx} = 0 \quad \Rightarrow \quad 3x^2 + 10x + 3 = 0$

$\Rightarrow (x+3)(3x+1) = 0 \Rightarrow x = -3, -\frac{1}{3}$

$\therefore$ tangents at $x = -3, -\frac{1}{3}$ are parallel to x-axis.

Also $\frac{dy}{dx} > 0$ if $(x+3)(3x+1) > 0$

i.e., if x does not lie between -3 and $-\frac{1}{3}$

$\therefore$ function is increasing in $(-\infty, -3) \cup \left(-\frac{1}{3}, \infty\right)$

Also $\frac{dy}{dx} < 0$ if $(x+3)(3x+1) < 0$

i.e., if x lies between -3 and $-\frac{1}{3}$

$\therefore$ function is decreasing in $\left(-3, -\frac{1}{3}\right)$

(*vii*) **Points of inflexion :** Now $\frac{d^2y}{dx^2} > 0$ when $6x + 10 > 0$ *i.e.*, $x > -\frac{5}{3}$

$\therefore$ graph of function is concave upwards for $x > -\frac{5}{3}$

Also $\frac{d^2y}{dx^2} < 0$ when $6x + 10 < 0$ *i.e.*, $x < -\frac{5}{3}$

$\therefore$ graph of function is concave downwards for $x < -\frac{5}{3}$

Again $\frac{d^2y}{dx^2}$ changes sign at $x = -\frac{5}{3}$

$\therefore$ graph of function has a point of inflexion at

$$x = -\frac{5}{3}, \quad y = -\frac{125}{27} + \frac{125}{9} - 5 - 4 = \frac{7}{27}$$

$\therefore \left(-\frac{5}{3}, \frac{7}{27}\right)$ is a point of inflexion.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College of Education Bakhtab Jammu

(*viii*) **Maxima and Minima**

Now $\frac{dy}{dx} = 0$ gives $x = -\frac{1}{3}, -3$

At $x = -\frac{1}{3}, \frac{d^2y}{dx^2} = -\frac{6}{3} + 10 = 8 > 0$

$\therefore$ function has a local minima at $x = -\frac{1}{3}$

and this local minimum value $= -\frac{1}{27} + \frac{5}{9} - 1 - 4 = -\frac{121}{27}$

At $x = -3, \frac{d^2y}{dx^2} = -18 + 10 = -8 < 0$

$\therefore$ function has a local maxima at $x = -3$

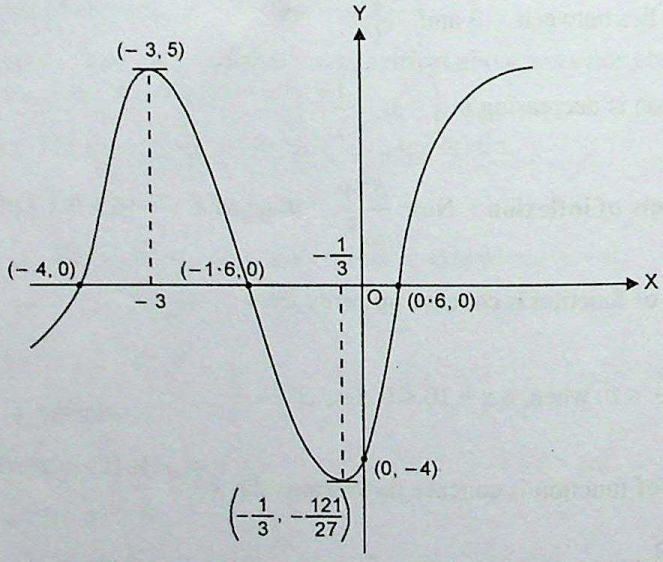
and this local maximum value $= -27 + 45 - 9 - 4 = 5$

(*ix*) **Additional Points**

Curve passes through $(1, 5), (2, 30), (-1, -3), (-2, 2), (-3, 5)$

Also $y \to \infty$ as $x \to \infty$ and also $y \to -\infty$ as $x \to -\infty$.

A rough sketch of the curve is given in the figure.



Example 3. Trace the curve $x = (y-1)(y-2)(y-3)$.

Sol. The equation of the curve is $x = (y-1)(y-2)(y-3)$ ...(1)

(*i*) **Symmetry :** The curve is neither symmetrical about axes nor about origin. Also the curve is neither symmetrical about $y = x$ nor about $y = -x$.

(*ii*) **Origin :** The curve does not pass through the origin.

(*iii*) **Points of intersection with axes :** The curve meets x-axis where $y = 0$

$\therefore$ putting $y = 0$ in (1), we get, $x = -6$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$\therefore$ curve meets x-axis in $(-6, 0)$

The curve meets y-axis where $x = 0$

$\therefore$ putting $x = 0$ in (1), we get, $(y-1)(y-2)(y-3) = 0$

$\therefore$ $y = 1, 2, 3.$

$\therefore$ curve meets y-axis in $(0, 1), (0, 2), (0, 3)$.

(*iv*) **Asymptotes :** The curve has no asymptotes.

(*v*) **Tangents :**

Now $x = y^3 - 6y^2 + 11y - 6$

$$\therefore \quad \frac{dx}{dy} = 3y^2 - 12y + 11$$

$$\therefore \quad \frac{dx}{dy} = 0 \text{ gives } 3y^2 - 12y + 11 = 0$$

$$\therefore \quad y = \frac{12 \pm \sqrt{144 - 132}}{6} = \frac{12 \pm 2\sqrt{3}}{6} = \frac{6 \pm 1{\cdot}732}{3} = 2{\cdot}6 \text{ (nearly)}, 1{\cdot}4 \text{ (nearly)}$$

When $y = 2{\cdot}6$, $x = -0{\cdot}384$ (nearly)

When $y = 1{\cdot}4$, $x = 0{\cdot}384$ (nearly)

$\therefore$ tangents to the curve at $(-0{\cdot}384, 2{\cdot}6)$ and $(0{\cdot}384, 1{\cdot}4)$ are parallel to the y-axis.

(*vi*) **Additional Points**

Now $y < 0 \quad \Rightarrow \quad x < 0$

$\therefore$ no portion of the curve lies in the fourth quadrant.

$0 < y < 1 \quad \Rightarrow \quad x < 0$

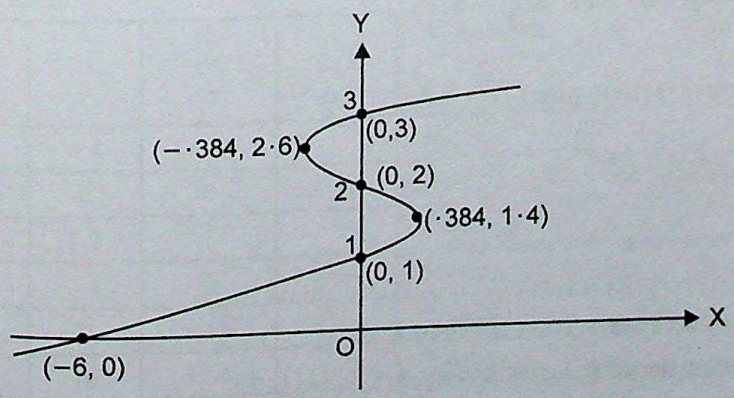
$1 < y < 2 \quad \Rightarrow \quad x > 0$

$2 < y < 3 \quad \Rightarrow \quad x < 0$

$3 < y \quad \Rightarrow \quad x > 0$

$x \to \infty \quad \Rightarrow \quad y \to \infty$

A rough sketch of the curve is given in the figure.



CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Education Bakhtalab Jammu

Example 4. Let $y = (x+1)^2 (x-2)$

Find the intervals of values of x for which the curve is

(*i*) rising (*ii*) falling (*iii*) concave upwards (*iv*) concave downwards.

Sketch the curve showing points of inflexion and asymptotes, if any.

Sol. 1. Symmetry. The curve is neither symmetrical about any of the axes nor about the origin.

2. Origin. The curve does not pass through the origin (0, 0)

3. Asymptotes. Curve has an equation of degree 3.

Co-efficient of x^3 is $1 (\neq 0)$, which is constant

$\therefore$ there is no asymptote parallel to x-axis.

Co-efficient of y is $-1 (\neq 0)$, which is constant

$\therefore$ there is no asymptote parallel to y-axis, $\phi_3(m) = 1$ (being constant), does not give any value of m.

$\therefore$ there is no oblique asymptote.

4. Points of intersection with the co-ordinate axes.

The curve cuts x-axis where $y = 0$

Putting $y = 0$, we get, $x = -1, 2$.

$\therefore$ curve passes through $(-1, 0)$ and $(2, 0)$

The curve cuts y-axis where $x = 0$

Putting $x = 0$, we get, $y = (0+1)^2 (0-2) = -2$

$\therefore$ the curve passes through $(0, -2)$.

5. Domain of definition. y exists for all real values of x.

So there is no end to the values of x and y.

6. $y = (x+1)^2 (x-2)$

$$\frac{dy}{dx} = 2(x+1)(x-2) + (x+1)^2 = (x+1)(2x-4+x+1) = (x+1)(3x-3)$$

$$= 3(x+1)(x-1) = 3(x^2-1)$$

$$\frac{d^2y}{dx^2} = 6x \text{ and } \frac{d^3y}{dx^3} = 6$$

The curve is rising where $\dfrac{dy}{dx} > 0$

i.e., $3(x^2-1) > 0$

i.e., $x^2 > 1$

i.e., $|x|^2 > 1$

i.e., $|x| > 1$

i.e., $x > 1$ or $x < -1$

Therefore the curve is rising in $(-\infty, -1) \cup (1, \infty)$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

The curve is falling where $\frac{dy}{dx} < 0$.

i.e., $|x| < 1$ (as above)

i.e., $-1 < x < 1$

Therefore the curve is falling in $(-1, 1)$.

Now $\frac{dy}{dx} = 0$ if $x = \pm 1$

When $x = 1$ $\frac{d^2y}{dx^2} = +\text{ve}$

$\therefore$ the curve has a local minima at $(1, -4)$

When $x = -1$, $\frac{d^2y}{dx^2} = -\text{ve}$

$\therefore$ the curve has a local maxima at $(-1, 0)$

Now $\frac{d^2y}{dx^2} > 0$

$\Rightarrow$ if $6x > 0$

$\Rightarrow$ if $x > 0$

$\therefore$ the curve is concave upwards on $(0, \infty)$

and $\frac{d^2y}{dx^2} < 0$ $\Rightarrow$ if $6x < 0$ $\Rightarrow$ if $x < 0$

$\therefore$ the curve is convex upwards on $(-\infty, 0)$

$$\frac{d^2y}{dx^2} = 0 \text{ when } x = 0$$

Further $\frac{d^2y}{dx^2}$ changes sign at $x = 0$

Therefore the curve has a point of inflexion at $(0, -2)$.

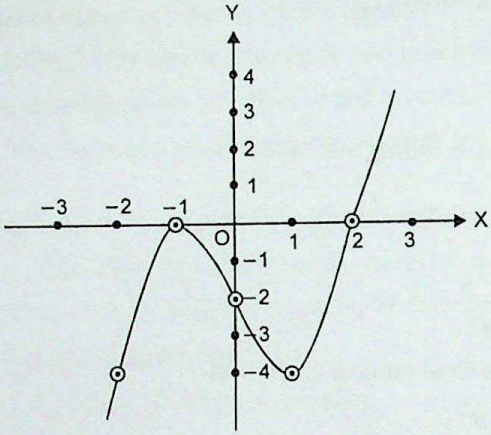
7. Table

x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	Remarks
-2	-4	$+$ve		Rising
-1	0	0	$-$ve	maxima, splits water, convex upwards
0	-2			falling, point of inflexion
1	-4	0	$+$ve	minima, holds water, concave upwards
2	0	$+$ve		Rising
3	16	$+$ve		Rising

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantala, Jaipur

8. Sketch



Example 5. Sketch the curve $y = x + \dfrac{1}{x}$. (J.U. 2016)

Sol. The equation of the curve is $y = x + \dfrac{1}{x}$ ...(1)

1. Symmetry. The graph is symmetrical with respect to the origin

if $x < 0$ then $y < 0$

and if $x > 0$ then $y > 0$

So, the graph of the function lies in the first and third quadrant only.

2. Origin. Origin does not lie on the graph.

3. Asymptotes. The equation of the curve is

$$x^2 - xy + 1 = 0 \qquad [F_2 + F_0 = 0]$$

Asymptotes are given by

$$x(x - y) = 0 \qquad \text{[By Inspection method]}$$

$x = 0$ and $y = x$ are the asymptotes.

4. Points of intersection with the co-ordinate axes

The curve meets the x-axis where $y = 0$

where $x^2 + 1 = 0 \quad \Rightarrow x = \pm i$ which are imaginary.

$\therefore$ the curve does not meet x-axis.

The curve meets y-axis where $x = 0$ which is impossible because Domain of definition of the graphs is all reals except $x = 0$. The curve does not meet y-axis. In fact y-axis is the asymptote of the curve.

5. Rising and falling

$$y = x + \frac{1}{x}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\therefore \quad \frac{dy}{dx} = 1 - \frac{1}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{2}{x^3}$$

The curve is rising where $\frac{dy}{dx} > 0$

$$\Rightarrow \quad 1 - \frac{1}{x^2} > 0$$

$$\Rightarrow \quad 1 > \frac{1}{x^2} \quad \Rightarrow \quad 1 < x^2 \quad \Rightarrow x^2 > 1 \quad \Rightarrow \quad |x|^2 > 1$$

$$\Rightarrow \quad |x| > 1 \quad \Rightarrow \quad x > 1 \quad \text{or} \quad x < -1$$

$\therefore$ the curve is rising $(-\infty, -1) \cup (1, \infty)$

The curve is falling where in

$$\frac{dy}{dx} < 0 \quad \Rightarrow \quad |x| < 1 \quad \Rightarrow \quad -1 < x < 1$$

$\therefore$ the curve is falling in $(-1, 1)$

$$\frac{dy}{dx} = 0 \text{ when } x = 1 \text{ or } x = -1$$

When $x = 1$, $\frac{d^2y}{dx^2} = +\text{ve}$

$\therefore$ the curve has a local minima at $(1, 2)$

When $x = -1$, $\frac{d^2y}{dx^2} = -\text{ve}$

$\therefore$ the curve has a local maxima at $(-1, -2)$

The curve is concave upwards if $\frac{d^2y}{dx^2} > 0$

$$\Rightarrow \quad \frac{2}{x^3} > 0 \quad \Rightarrow x > 0$$

$\therefore$ the curve is concave upwards on $(0, \infty)$

Similarly the curve is concave downwards on $(-\infty, 0)$.

Now $\frac{d^2y}{dx^2} \neq 0 \ \forall \ x$

Hence there is no point of inflexion.

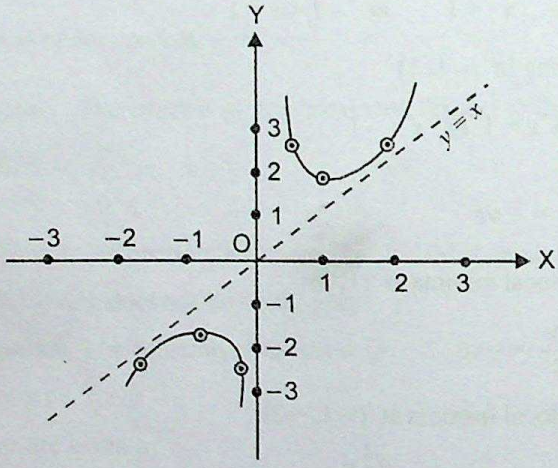
x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	Remarks
-2	$-\frac{5}{2}$	$+$ve		Rising

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$-\frac{3}{2}$	$-\frac{13}{6}$	+ ve		Rising
-1	-2	0	– ve	maxima, splits water, convex upwards
$-\frac{1}{2}$	$-\frac{5}{2}$	– ve		falling
$\frac{1}{2}$	$\frac{5}{2}$	– ve		falling
1	2	0	+ ve	minima, holds water, concave upwards
$\frac{3}{2}$	$\frac{13}{6}$	+ ve		rising
2	$\frac{5}{2}$	+ ve		rising

The curve is



Example 6. Trace the curve $y = \dfrac{x}{1+x^2}$.

Sol. The equation of the curve is

$$y = \frac{x}{1+x^2} \quad \text{or} \quad (1+x^2)\, y - x = 0 \qquad \ldots(1)$$

(*i*) It is not symmetrical about any axis. Also it is not symmetrical about origin.

(*ii*) The curve passes through the origin and the equation of the tangent at origin is $y - x = 0$.

(*iii*) Domain is $(-\infty, \infty)$.

The equation of curve is $y x^2 - x + y = 0$

$\because$ x is real, $\quad \therefore \; (-1)^2 - 4y^2 \geq 0$

$\Rightarrow 4y^2 \leq 1 \quad \Rightarrow y^2 \leq \frac{1}{4} \quad \Rightarrow |y|^2 \leq \frac{1}{4}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College of Education Bantalab Jammu

$\Rightarrow \ |y| \le \frac{1}{2} \quad \Rightarrow \ -\frac{1}{2} \le y \le \frac{1}{2}$

$\therefore$ max. value of y is $\frac{1}{2}$ and minimum value of y is $-\frac{1}{2}$.

(*iv*) curve meets both the axes in $(0, 0)$ only.

(*v*) $y = 0$ is the asymptote of the curve.

Also $x > 0 \Rightarrow y > 0$, so the curve lies above the asymptote for $x > 0$. Again as $x < 0 \Rightarrow y < 0$, so the curve lies below the asymptote for $x < 0$.

(*vi*) $$\frac{dy}{dx} = \frac{(1+x^2).1 - x.2x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(1+x^2)^2.(-2x) - (1-x^2).2(1+x^2).2x}{(1+x^2)^4}$$

$$= \frac{-2x - 2x^3 - 4x + 4x^3}{(1+x^2)^3} = \frac{2x(x^2-3)}{(1+x^2)^3}$$

Now $\frac{dy}{dx} = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$\therefore$ tangents at $x = -1, 1$ are parallel to x-axis.

At $x = -1$, $\frac{d^2y}{dx^2} = \frac{2(-1)(1-3)}{(1+1)^3} = \frac{1}{2} > 0$

$\therefore$ curve has a local minima at $x = -1$.

At $x = 1$, $\frac{d^2y}{dx^2} = \frac{2(1)(1-3)}{(1+1)^3} = -\frac{1}{2} < 0$

$\therefore$ curve has a local maxima at $x = 1$.

Now $\frac{dy}{dx} > 0$ for $\frac{1-x^2}{(1+x^2)^2} > 0$ *i.e.*, for $x^2 < 1$

i.e., for $|x|^2 < 1$ *i.e.*, for $|x| < 1$

i.e., for $-1 < x < 1$

$\therefore$ curve is increasing for $-1 < x < 1$.

Also curve is decreasing for $x < -1$ and $x > 1$.

Again $\frac{d^2y}{dx^2} = \frac{2x(x^2-3)}{(1+x^2)^3} \Rightarrow \frac{d^2y}{dx^2} = 0$ when $x(x^2-3) = 0$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

i.e., when $x = 0, \sqrt{3}, -\sqrt{3}$ and $\dfrac{d^2y}{dx^2}$ changes sign at these values of x.

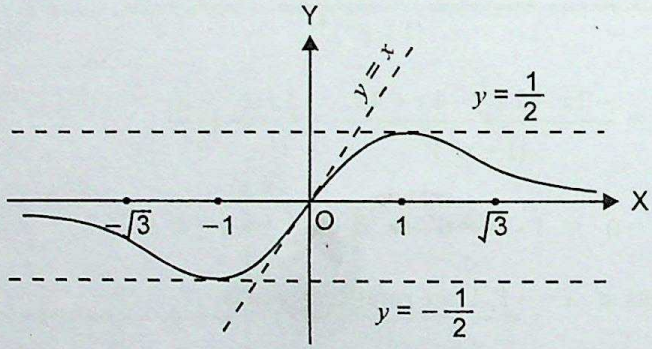
$\therefore$ curve has points of inflexion at $x = (0, 0), \left(\sqrt{3}, \dfrac{\sqrt{3}}{4}\right)$ and $\left(-\sqrt{3}, -\dfrac{\sqrt{3}}{4}\right)$

Again $\dfrac{d^2y}{dx^2} > 0$ when $-\sqrt{3} < x < 0$ or $x > \sqrt{3}$

$\therefore$ curve is concave upwards in $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

Similarly curve is concave downwards in $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

A rough sketch of the curve is shown in the figure.



Example 7. Trace the curve $y = \dfrac{x^3}{1+x^2}$.

Sol. The equation of the curve is

$$y = \frac{x^3}{1+x^2} \quad \text{or} \quad y(1+x^2) - x^3 = 0 \qquad \ldots(1)$$

(*i*) The curve is not symmetrical about any axis.

(*ii*) The curve passes through the origin and tangent at origin is $y = 0$ *i.e.*, x axis.

(*iii*) x can take any real value.

Also $y = \dfrac{x^3}{1+x^2} = x\left(\dfrac{x^2}{1+x^2}\right)$

$\therefore$ $x > 0 \Rightarrow y > 0$ and $x < 0 \Rightarrow y < 0$.

Therefore curve lies in the first and third quadrants only.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College of Education Bantalab Jammu

(*iv*) Equation (1) can be written as

$$y = \frac{x^3}{1+x^2} = \frac{x^3}{x^2\left(1+\frac{1}{x^2}\right)} = x\left(1+\frac{1}{x^2}\right)^{-1} = x\left(1-\frac{1}{x^2}+\frac{1}{x^4}\ldots\right)$$

or $\quad y = x - \frac{1}{x} + \ldots$

$\therefore \quad y = x$ is an asymptote of the curve and the curve lies below the asymptote for $x > 0$ and above the asymptote when $x < 0$.

[Note. (1) If the equation of the curve can be put in the form $y = mx + c + \frac{\alpha}{x} + \frac{\beta}{x^2} + \ldots$, then $y = mx + c$ is an asymptote of the curve.

(2) The curve lies above or below the asymptote according as $\alpha > 0$ or $\alpha < 0$. If $\alpha = 0$, then the curve lies above or below the asymptote according as $\beta > 0$ or $\beta < 0$]

(*v*) $$\frac{dy}{dx} = \frac{(1+x^2)\cdot 3x^2 - x^3\cdot 2x}{(1+x^2)^2} = \frac{3x^2+x^4}{(1+x^2)^2} = \frac{x^2(x^2+3)}{(1+x^2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(1+x^2)^2\cdot(6x+4x^3) - (3x^2+x^4)\cdot 2(1+x^2)\cdot 2x}{(1+x^2)^4} = \frac{2x(3-x^2)}{(1+x^2)^3}$$

$\frac{dy}{dx} > 0$ for $\frac{x^2(x^2+3)}{(1+x^2)^2} > 0$ *i.e.,* for all x.

$\therefore$ curve is increasing for all x

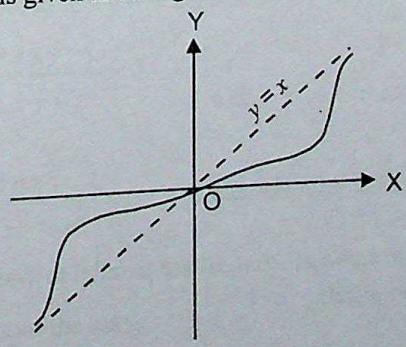
Again $\frac{d^2y}{dx^2} = \frac{2x(3-x^2)}{(1+x^2)^3} > 0$ for $x(3-x^2) > 0$

i.e., for $x \in (-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

$\therefore$ curve is concave upwards in $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

and hence concave downwards in $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

A rough sketch of the is given in the figure.



CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantala, Jadavpur

Example 8. Trace the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$.

Sol. The equation of the curve is $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$ ...(1)

(*i*) **Symmetry :** Here powers of x as well y are even.

$\therefore$ curve is symmetrical about both the axes.

(*ii*) **Origin :** The curve does not pass through the origin.

(*iii*) **Points of intersection with axes :** The curve meets x-axis where $y = 0$.

$\therefore$ putting $y = 0$ in (1), we get,

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} = 1 \quad \text{or} \quad \left(\frac{x}{a}\right)^{2} = 1$$

$\therefore$ $x^2 = a^2$ or $x = \pm a$

$\therefore$ curve meets x-axis in $(-a, 0)$, $(a, 0)$.

Again the curve meets y-axis where $x = 0$

$\therefore$ putting $x = 0$ in (1), we get, $y = \pm b$

$\therefore$ curve meets y-axis in $(0, -b)$, $(0, b)$.

(*iv*) **Asymptotes :** The curve has no asymptotes.

(*v*) **Tangents :** Differentiating both sides of (1) w.r.t. x,

$$\frac{2}{3}\left(\frac{x}{a}\right)^{-\frac{1}{3}} \cdot \frac{1}{a} + \frac{2}{3}\left(\frac{y}{b}\right)^{-\frac{1}{3}} \cdot \frac{1}{b}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{b\left(\frac{y}{b}\right)^{\frac{1}{3}}}{a\left(\frac{x}{a}\right)^{\frac{1}{3}}} = -\left(\frac{b^2 y}{a^2 x}\right)^{\frac{1}{3}}$$

Now $\frac{dy}{dx} = 0$ when $y = 0$, $x = \pm a$

Also $\frac{dy}{dx} \to \infty$ when $x = 0$, $y = \pm b$

$\therefore$ tangents to the curve at $(\pm a, 0)$ are parallel to x-axis and tangents to the curve at $(0, \pm b)$ are parallel to y-axis.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

(*vi*) **Region**

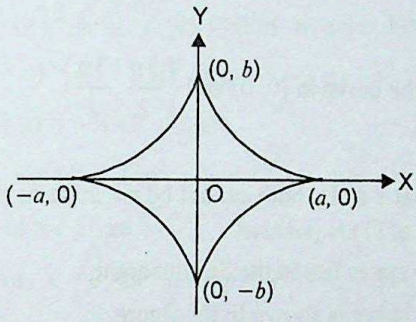
From (1), $\left(\frac{y}{b}\right)^{\frac{2}{3}} = 1 - \left(\frac{x}{a}\right)^{\frac{2}{3}}$

Now $|x| > a \quad \Rightarrow \quad y^2 < 0 \quad \Rightarrow y$ is imaginary.

$\therefore$ the curve does not lies beyond the lines $x = \pm a$.

Similarly the curve does not lie beyond the lines $y = \pm b$.

A rough sketch of the curve is given in the figure.



Example 9. Trace the curve $x^3 + y^3 = 3\,a\,x\,y,\ a \geq 0$.

Sol. The equation of the curve is $x^3 + y^3 = 3\,a\,x\,y,\ a \geq 0$ ...(1)

(*i*) **Symmetry :** The given equation (1) does not change when x is changed to y and y is changed to x.

curve is symmetrical about the line $y = x$.

(*ii*) **Origin :** The curve passes through the origin.

The tangents at origin are given by $x\,y = 0$

i.e., $x = 0, y = 0$. These tangents are different.

$\therefore$ origin is a node.

(*iii*) **Asymptotes :** (1) can be written as

$$(x + y)\ (x^2 - x\,y + y^2) - 3\,a\,x\,y = 0$$

Asymptote (if any) parallel to $x + y = 0$ is given by

$$x + y - \underset{\substack{x \to \infty \\ y = -x}}{\text{Lt}} \frac{3\,a\,x\,y}{x^2 - x\,y + y^2} = 0$$

or $\quad x + y - \underset{x \to \infty}{\text{Lt}} \dfrac{-3\,a\,x^2}{x^2 + x^2 + x^2} = 0 \quad$ or $\quad x + y + \underset{x \to \infty}{\text{Lt}} \dfrac{3\,a\,x^2}{3\,x^2} = 0$

or $\quad x + y + a = 0$

This is the only asymptote of the curve.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

(*iv*) **Points of intersection with axes**

Putting $x = 0$ in (1), we get $y = 0$

Putting $y = 0$ in (1), we get $x = 0$

$\therefore$ curve meets axes in (0, 0) only.

Putting $y = x$ in (1), we get,

$$x^3 + x^3 = 3\,a\,x^2 \quad \text{or} \quad x^2\,(2\,x - 3\,a) = 0$$

$$\therefore \qquad x = 0,\ \frac{3\,a}{2}$$

$$\therefore \qquad y = 0,\ \frac{3\,a}{2}$$

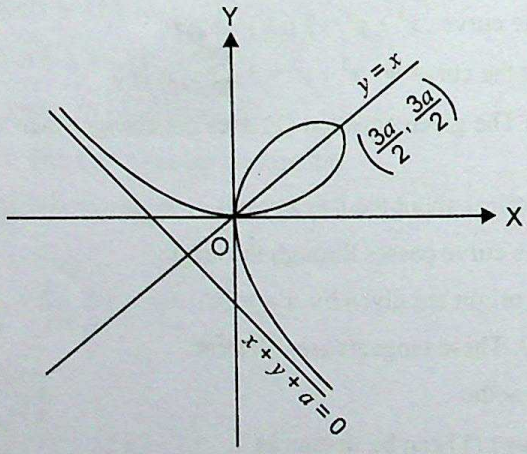
$\therefore$ line $y = x$ meets the curve in (0, 0) and $\left(\frac{3\,a}{2}, \frac{3\,a}{2}\right)$

(*v*) **Region**

From (1), it is clear that x and y both cannot be negative as in that case L.H.S. of (1) is negative whereas R.H.S. of (1) is positive.

$\therefore$ no portion of the curve lies in the 3rd quadrant.

A rough sketch of the curve is shown in the figure.



Example 10. Trace the curve $y^2\,(a + x) = x^2\,(3\,a - x),\ a > 0$.

Sol. The equation of the curve is $y^2\,(a + x) = x^2\,(3\,a - x),\ a > 0$ ...(1)

(*i*) **Symmetry.** In (1), only even powers of y occur

$\therefore$ curve is symmetrical about x-axis.

(*ii*) **Origin.** Curve passes through the origin

Tangents at origin are given by

$$a\,y^2 - 3\,a\,x^2 = 0 \qquad \text{[Equating to zero the lowest degree terms]}$$

or $y^2 - 3\,x^2 = 0$ or $y = \pm\sqrt{3}\,x$.

Therefore origin is a node as the two tangents at origin are real and different.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Sant Longowal Memorial College Of Education Bantalab Jammu

(*iii*) **Asymptotes.** The curve given by (1) has only one asymptote

$$x + a = 0 \qquad i.e.,\ x = -a.$$

(*iv*) **Points of intersection with axes :**

The curve meets x-axis where $y = 0$

Putting $y = 0$ in (1), we get,

$$x^2 (3a - x) = 0 \quad \Rightarrow \quad x = 0, 3a$$

$\therefore$ curve meets x-axis in $(0, 0)$, $(3a, 0)$

Shifting the origin to $(3a, 0)$ *i.e.*, by putting $x = X + 3a$, $y = Y + 0$ in (1), we get,

$$Y^2 (a + X + 3a) = (X + 3a)^2 (3a - X - 3a)$$

or $\quad Y^2 (X + 4a) = -X (X + 3a)^2$

or $\quad Y^2 (X + 4a) + X (X + 3a)^2 = 0$

Tangents at $(3a, 0)$ are given by

$$9a^2 X = 0 \quad \text{or } X = 0 \quad \text{or } x - 3a = 0 \quad \text{or } x = 3a$$

$\therefore$ tangent to the curve at $(3a, 0)$ is parallel to y-axis.

(*v*) **Region.** From (1), it is clear that x cannot be less than $-a$ as for $x < -a$, $y^2 (a + x)$ is negative and $x^2 (3a - x)$ is positive.

$\therefore$ no portion of the curve lies to the left of $x = -a$

Similarly no portion of the curve lies to the right of $x = 3a$.

$\therefore$ curve lies between the line $x = -a$ and $x = 3a$

(*vi*) **Tangents**

From (1), $y = x\sqrt{\dfrac{3a - x}{a + x}}$ (Taking + ve value only)

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left[\frac{x\sqrt{3a - x}}{\sqrt{a + x}}\right]$$

$$= \frac{\sqrt{a + x}\left[x.\dfrac{-1}{2\sqrt{3a - x}} + \sqrt{3a - x}.1\right] - x\sqrt{3a - x}\,\dfrac{1}{2\sqrt{a + x}}}{a + x}$$

$$= \frac{-\dfrac{x\sqrt{a + x}}{2\sqrt{3a - x}} + \dfrac{\sqrt{a + x}\sqrt{3a - x}}{1} - \dfrac{x\sqrt{3a - x}}{2\sqrt{a + x}}}{a + x}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$= \frac{-x(a+x)+2(a+x)(3a-x)-x(3a-x)}{2(a+x)^{\frac{3}{2}}\sqrt{3a-x}}$$

$$= \frac{-ax-x^2+6a^2+4ax-2x^2-3ax+x^2}{2(a+x)^{\frac{3}{2}}\sqrt{3a-x}}$$

$$= \frac{-2x^2+6a^2}{2(a+x)^{\frac{3}{2}}\sqrt{3a-x}} = \frac{3a^2-x^2}{(a+x)^{\frac{3}{2}}\sqrt{3a-x}}$$

Now $\frac{dy}{dx}=0 \quad \Rightarrow \quad 3a^2-x^2=0 \quad \Rightarrow \quad x=\pm\sqrt{3}\,a$

$\Rightarrow \quad x=\sqrt{3}\,a$ $\qquad$ [$\because$ $x=-\sqrt{3}\,a$ is impossible]

(*vii*) **Intervals of increase and decrease**

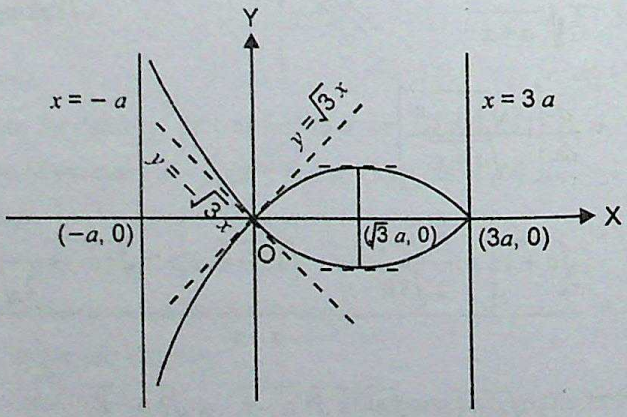
$\frac{dy}{dx}>0$ for $3a^2-x^2>0$ $\quad$ *i.e.*, for $|x|<\sqrt{3}\,a$

$\therefore$ the curve increases in $[0, \sqrt{3}\,a]$

Again $\frac{dy}{dx}<0$ for $3a^2-x^2<0$ $\quad$ *i.e.*, for $|x|>\sqrt{3}\,a$

$\therefore$ the curve decreases in $[\sqrt{3}\,a, 3a]$

A rough sketch of the curve is given in the figure.



CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Sandhi Memorial College Of Education Bantalab Jammu

EXERCISE 10 (a)

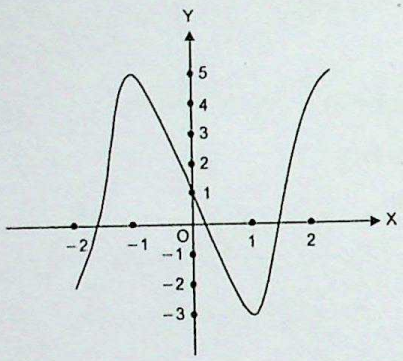
1. Trace the curve $y = 2x^3 - 6x + 1$
2. Trace the curve $y = x^3 - 3x^2 + 3$
3. Trace the curve is $y = \frac{1}{6}(x^3 - 6x^2 + 9x + 6)$
4. Trace the curve $y = x(x+1)(x-2)$.
5. Trace the curve $y = x^2(x - 3a)$, $a > 0$.
6. Trace the curve $y = (x+1)^2(x-3)$.
7. Trace the curve $y = x + \frac{4}{x}$.
8. Trace the curve $y = \frac{x^2+1}{x+1}$.
9. Trace the curve $y = -\frac{x^2}{1+x^2}$.
10. Trace the curve $y = \frac{x^2}{1+x^2}$.
11. Trace the curve $y = -\frac{x}{1+x^2}$.
12. Trace the curve $y = \frac{x^2}{x^2-1}$
13. Trace the curve $y = \frac{x}{x^2-1}$.
14. Trace the curve $y^2 = (x+1)^3$.
15. Trace the curve $y^2 = (x-2)(x-5)^2$.
16. Trace the curve $y^2 = x(x-a)^2$, $a > 0$.
17. Trace the curve $x(x^2+y^2) = a(x^2-y^2)$, $a > 0$.
18. Trace the curve $(a^2+x^2)y^2 = x^2(a^2-x^2)$
19. Trace the curve $ay^2 = x^2(a-x)$, $a > 0$.
20. Trace the curve $x^2y^2 = x^2 - a^2$
21. Trace the curve $x^2y^2 = a^2(y^2-x^2)$
22. Trace the curve $y^2(x^2-4) = x^4$
23. Trace the curve $x^3 + y^3 = ax^2$, $a > 0$.
24. Trace the curve $x^3 + y^3 = a^2x$.
25. Trace the curve $x(x^2+y^2) = 2ay^2$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

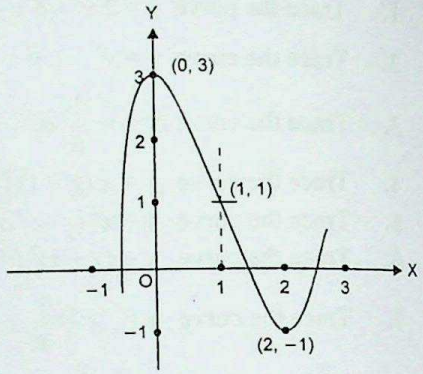
Gandhi Memorial College Of Education Bantalab Jammu

ANSWERS

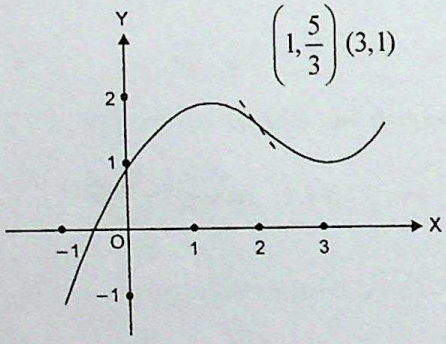
1.



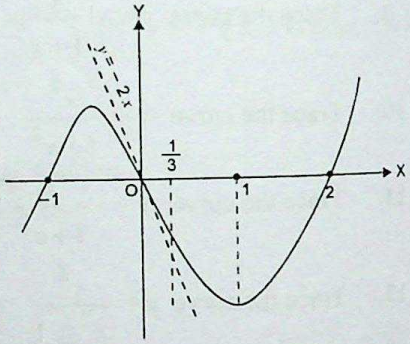
2.



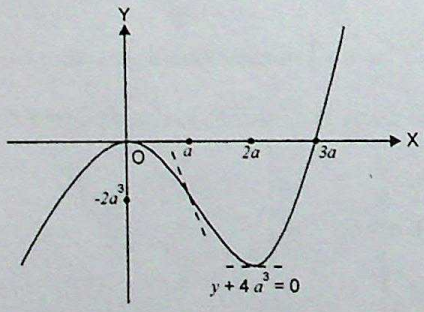
3.



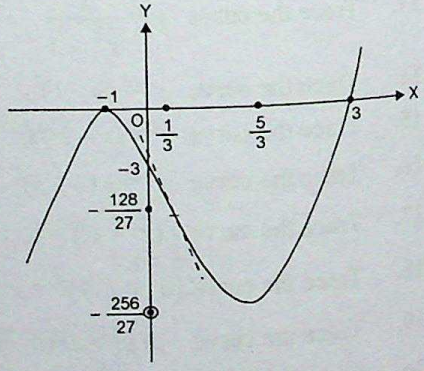
4.



5.



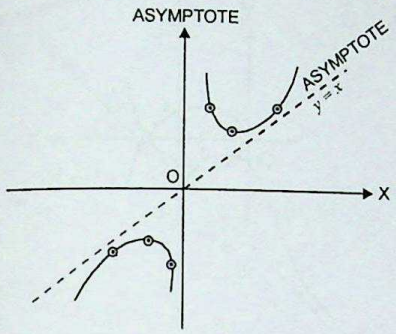
6.



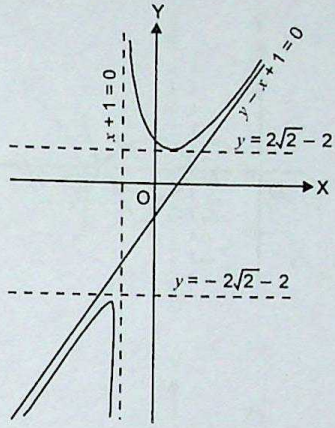
CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Education Bantalab Jammu

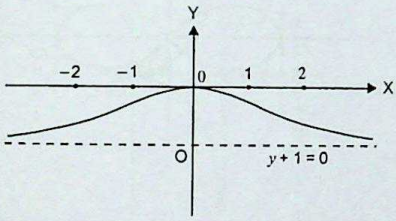
7.



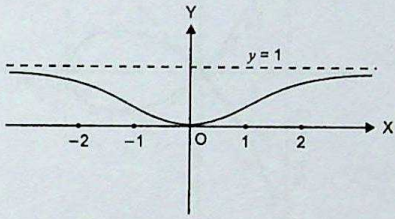
8.



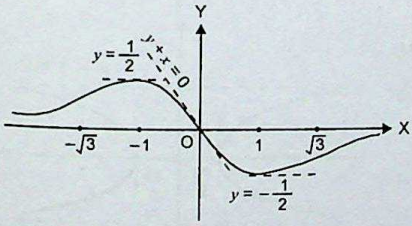
9.



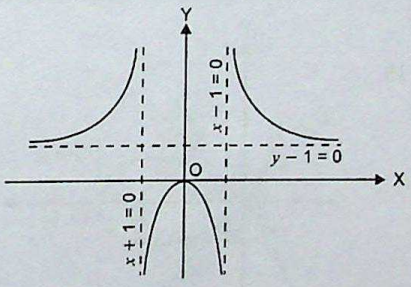
10.



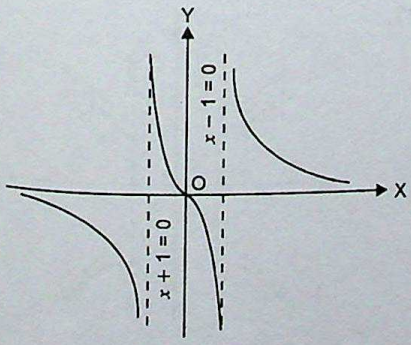
11.



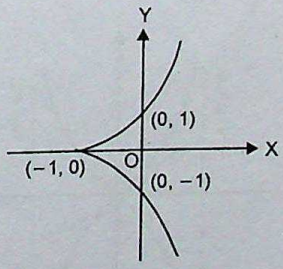
12.



13.



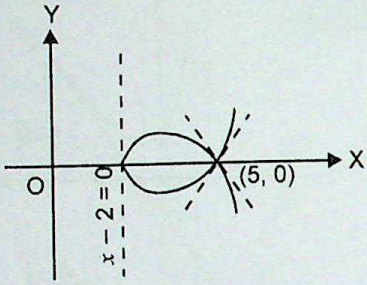
14.



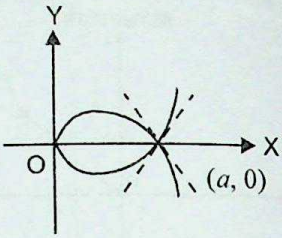
CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

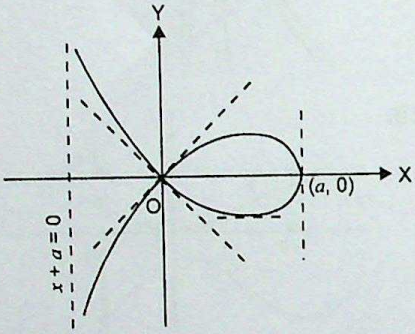
15.



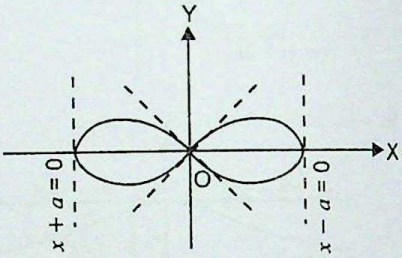
16.



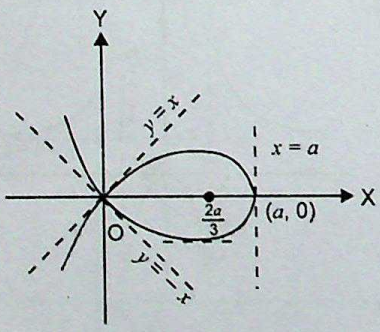
17.



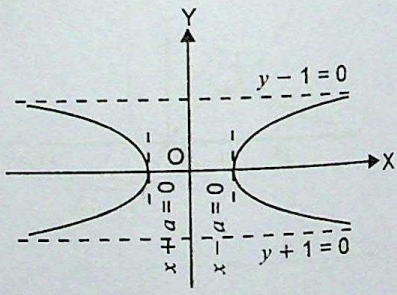
18.



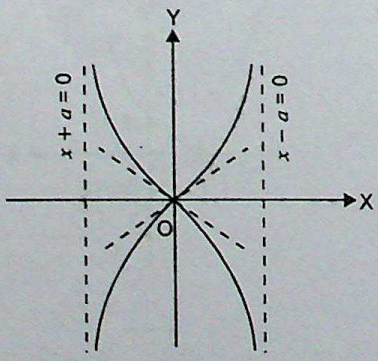
19.



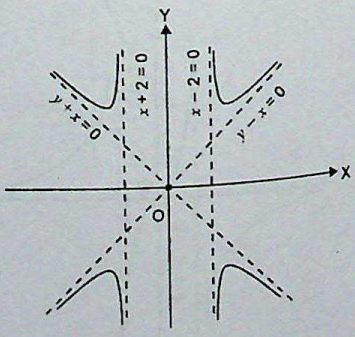
20.



21.



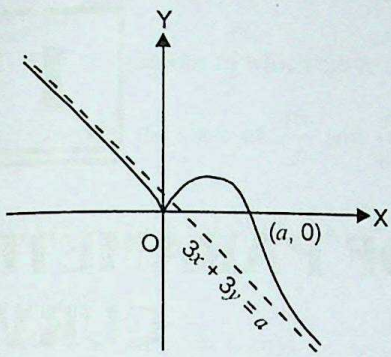
22.



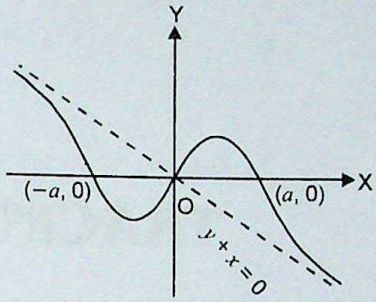
CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

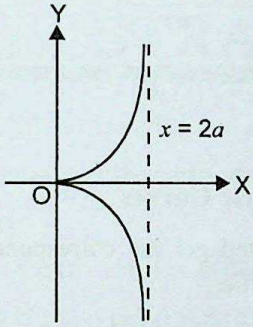
23.



24.



25.



CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

11

TRACING OF PARAMETRIC CURVES

Art-1. Parametric Representation of Curves

Parametric Representation of Cartesian Curve $y = f(x)$ is

$x = f(t)$ and $y = \phi(t)$ where t is a parametre.

Art-2. Working Method for Tracing Parametric Curves

Case I. Eliminate the parameter if possible and get the corresponding cartesian equation of the curve which can be traced as done earlier.

Case II. If the parameter cannot be easily eliminated from the given equations, then we proceed like this :

I. Symmetry

(*i*) If $x = f(t)$ is an even function of t and $y = \phi(t)$ an odd function of t, then the curve is symmetrical about x-axis.

(*ii*) If $x = f(t)$ is an odd function of t and $y = \phi(t)$ an even function of t, then curve is symmetrical about y-axis.

(*iii*) If $x = f(t)$ and $y = \phi(t)$ are both odd functions of t, then the curve is symmetrical in opposite quadrants.

II. Origin

If by putting $x = 0$, we get a real value of t, which makes y equals zero, then the curve passes through the origin.

III. Axes Intersection

Find the points of intersection of the curve and coordinate axes.

IV. Limitations

If possible, find the greatest and the least values of x and y which give us lines parallel to axes between which the curve lies or does not lie.

V. Points

Find the points where $\frac{dy}{dx} = 0$, $\frac{dy}{dx} \to \infty$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

VI. Region

(*i*) Find the regions in which curve does not lie.

(*ii*) Consider the signs of $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

(*iii*) Consider the values of $x, y, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dy}{dx}$.

VII. Asymptotes

Find the asymptotes, if any.

ILLUSTRATIVE EXAMPLES

Example 1. Trace the curve $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

Sol. The equations of the curve are $x = a(t - \sin t)$, $y = a(1 - \cos t)$

I. Symmetry

Here $x = a(t - \sin t)$ is an odd function of t

and $y = a(1 - \cos t)$ is an even function of t

$\therefore$ curve is symmetrical about y-axis.

II. Origin

Now $x = 0 \Rightarrow a(t - \sin t) = 0 \Rightarrow t - \sin t = 0$

$\Rightarrow$ $t = 0$ and then $y = a(1 - 1) = 0$

$\therefore$ curve passes through the origin.

III. Axis Intersection

(*i*) The curve cuts the x-axis where

$y = 0$ *i.e.,* $a(1 - \cos t) = 0 \Rightarrow \cos t = 1 \Rightarrow t = 0, 2\pi$

$\therefore$ $x = 0, 2a\pi$

$\therefore$ curve cuts x-axis at $(0, 0), (2a\pi, 0)$

(*ii*) The curve cuts y-axis where

$x = 0$ *i.e.,* $a(t - \sin t) = 0 \Rightarrow \sin t = t$

$\Rightarrow$ $t = 0$ and then $y = a(1 - 1) = 0$

$\therefore$ curve cuts y-axis at $(0, 0)$

IV. Limitations

The greatest value of y is $2a$ and least value is 0. Hence curve lies between the lines $y = 0$ and $y = 2a$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College of Education Bantalab Jammu

V. Points

$$\frac{dx}{dt} = a(1-\cos t), \quad \frac{dy}{dt} = a \sin t$$

$$\therefore \quad \frac{dy}{dx} = \frac{a \sin t}{a(1-\cos t)} = \frac{\sin t}{1-\cos t} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}} = \cot\frac{t}{2}$$

$$\therefore \quad \frac{dy}{dx} = 0 \quad \text{when} \quad t = \pi$$

Also $\frac{dy}{dx} \to \infty$ when $t = 0, 2\pi$

VI. Region

(*i*) Now $y = a(1-\cos t) = 2a\sin^2\frac{t}{2}$

$$\therefore \quad \sin\frac{t}{2} = \sqrt{\frac{y}{2a}}$$

Now $\sin\frac{t}{2}$ is imaginary for negative values of y

$\therefore$ no portion of the curve lies below x-axis.

Also $\sin^2\frac{t}{2} \le 1$

$\therefore$ no portion of the curve lies above the line for which $y > 2a$.

(*ii*) As x increases from 0 to $a\pi$, y also increases for 0 to $2a$.

When x increases from $a\pi$ to $2a\pi$, y decreases from $2a$ to 0.

VII. Asymptotes

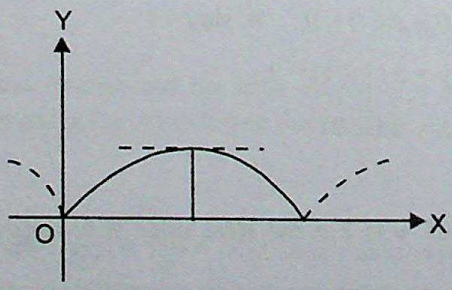
When $t \to \infty$, $x \to \infty$, then y does not tend to a limit.

$\therefore$ curve has no asymptote parallel to x-axis.

Also y cannot tend to infinity

$\therefore$ curve has no asymptote parallel to y-axis.

From the facts gathered above, shape of the curve is shown in the figure given below :



CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 2. Trace the curve

$$x = a(\theta + \sin\theta); \quad y = a(1 + \cos\theta), -\pi \le \theta \le \pi \qquad \text{[Cycloid]}$$

Sol. The equations of the curve are

$$x = a(\theta + \sin\theta), y = a(1 + \cos\theta)$$

Here the parameter θ cannot be easily eliminated.

1. Symmetry. The curve is symmetrical about the axis of y for $(\theta + \sin\theta)$ is an odd function of θ and $(1 + \cos\theta)$ is an even function of θ.

2. Origin. The curve does not pass through the origin.

3. Intercepts. It meet the x-axis when

$$y = 0 \quad i.e., \quad 1 + \cos\theta = 0$$

or $\cos\theta = -1$ *i.e.*, $\theta = \pi, -\pi$

$\therefore$ the points of intersection with the x-axis are $A(a\pi, 0)$, $A'(-a\pi, 0)$. Again it meets the y-axis when $x = 0$

i.e., $\theta + \sin\theta = 0$ or $\sin\theta = -\theta$ or $\theta = 0$

$\therefore$ it meets the axis of y at $B(0, 2a)$.

4. Asymptotes. There are no asymptotes.

5. Points. We have $\dfrac{dx}{d\theta} = a(1 + \cos\theta)$; $\dfrac{dy}{d\theta} = -a\sin\theta$

$$\therefore \quad \frac{dy}{dx} = \frac{-a\sin\theta}{a(1+\cos\theta)} = -\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = -\tan\frac{\theta}{2}$$

$$\therefore \quad \frac{dy}{dx} = 0 \text{ when } \theta = 0$$

i.e., at $(0, 2a)$, the tangent is parallel to the axis of x.

Also $\dfrac{dy}{dx} \to \infty$ when $\theta = \pi, -\pi$

$\therefore$ at $(a\pi, 0)$ and $(-a\pi, 0)$, the tangent is perpendicular to the axis of x.

6. Region. For all values of θ, $\dfrac{dx}{d\theta}$ is + ve

$\therefore$ x always increases with θ.

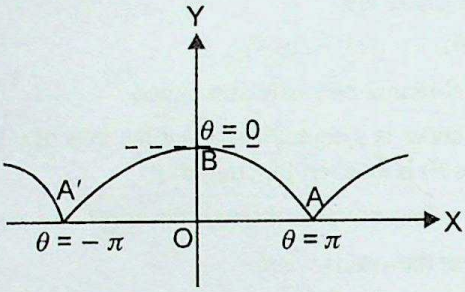
Also $\dfrac{dy}{d\theta}$ is + ve for $-\pi \le \theta \le \pi$.

Hence y increases when θ increases from $-\pi$ to 0 and y decreases when θ increases from 0 to π.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Hence approximately, the shape of the curve is as shown in the diagram.

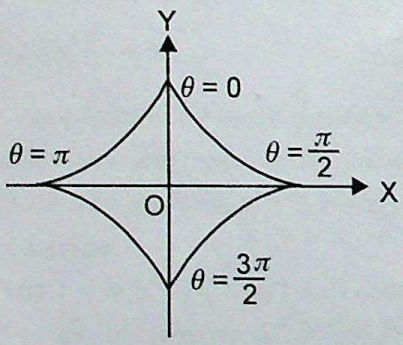


EXERCISE 11 (a)

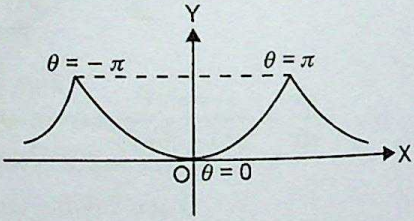
1. Trace the curve $x = a\cos^3\theta$; $y = a\sin^3\theta$ [Astroid]
2. Trace the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$
3. Trace the curve $x = a\left(\cos t + \frac{1}{2}\log\tan^2\frac{t}{2}\right)$, $y = a\sin t$. [Tractrix]

ANSWERS

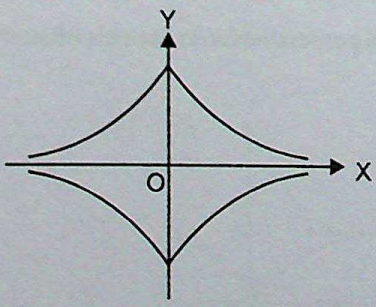
1.



2.



3.



CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

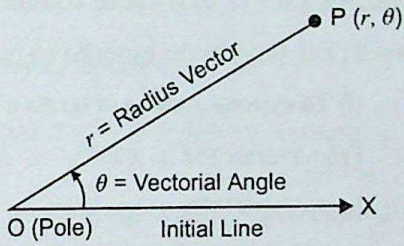
Gandhi Memorial College Of Education Bantalab Jammu

12

POLAR CO-ORDINATES

Art-1. Polar Co-ordinates

Let O be a fixed point and OX a fixed straight line through it, whose positive direction OX as shown by the arrow. O is called the **pole**, and OX is called the **initial line.**



Let P be a point in a plane through the initial line. Join OP. Then

(*i*) the length OP is called the **radius vector** of P, and is denoted by r,

(*ii*) the angle XOP is called the **vectorial angle** of P, and is denoted by θ, and

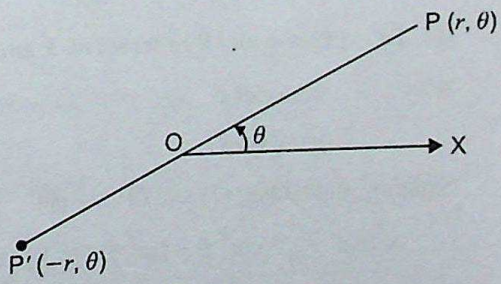
(*iii*) the two together , taken in this particular order, are called the **polar co-ordinates** of P, and are denoted by (r, θ).

Signs of the co-ordinates

(*i*) θ is regarded as positive, if it is traced in the counter-clockwise direction, and is negative, if it is traced in the clock-wise direction. Therefore, it follows that (r, θ) and $(r, \theta + 2\,n\,\pi)$, where $n \in \mathbf{Z}$, represent the same point and consequently, the vectorial angle can have infinitely many values, with the same value of r .

On this account, we say that *vectorial angle of a point is not unique.*

(*ii*) 'r', the radius vector, is positive, if it is cut along the line bounding the vectorial angle and is negative, if it is cut along the opposite direction of the line bounding the vectorial angle.



Let P be the point having the co-ordinates (r, θ).

Produce PO to P′, such that O is mid-point of PP′ *i.e.*, $|OP| = |OP'|$; then P′ will have polar co-ordinates $(-r, \theta)$.

Note : OP′ is a bounding line of vectorial angle $\theta + \pi$ or $\theta - \pi$

$\therefore$ polar co-ordinates of P′ are also $(r, \theta \pm \pi)$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

For vectorial angles $\theta + \pi$, or $\theta - \pi$, OP is negative,

$\therefore$ co-ordinates of P are $(-r, \theta \pm \pi)$.

$\therefore$ we conclude that, the polar co-ordinates of a point are *not unique*, whereas, the rectangular cartesian co-ordinates of a point are *unique*.

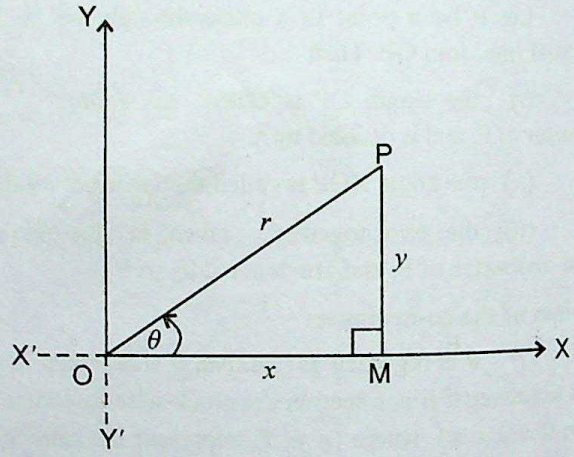
Conclusion : Giving due consideration to the signs of r and θ, we conclude that the general co-ordinates of a point (r, θ) are $((-1)^n r, \theta + n\pi)$; where n is any integer.

Art-2. Relation between Rectangular and Polar Co-ordinates

Given the point P (x, y) in Cartesian Co-ordinate system, express x and y in Polar form. Also, if P (r, θ) is a polar coordinate form, express r, θ in Carteisan form.

Proof : Let (x, y) be the cartesian co-ordinates of P, and (r, θ) be its polar co-ordinates.

(*i*) To express x and y in terms of r and θ



From P draw PM $\perp$ X'OX.

In *rt.* $\angle d$ ΔOMP,

$$\frac{OM}{OP} = \cos\theta$$

$\therefore$ $OM = OP \cos\theta$

$\therefore$ $x = r\cos\theta$

and $$\frac{MP}{OP} = \sin\theta$$

$\therefore$ $MP = OP \sin\theta$

$\therefore$ $y = r\sin\theta$

$\therefore$ we have $x = r\cos\theta$, $y = r\sin\theta$.

(*ii*) To express r and θ in terms of x and y

Now $x = r\cos\theta$ ...(1)

$y = r\sin\theta$ ...(2)

Squaring and adding (1) and (2), we get

$$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2(1) = r^2$$

$\therefore$ $r^2 = x^2 + y^2$, or $r = \sqrt{x^2 + y^2}$

Dividing (2) by (1), we get

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$\therefore \quad \tan\theta = \dfrac{y}{x}$, or $\theta = \tan^{-1}\dfrac{y}{x}$

$\therefore$ we have $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\dfrac{y}{x}$.

Note 1. In practice we generally use the relations :

$$r = \sqrt{x^2 + y^2}, \cos\theta = \frac{x}{r}, \sin\theta = \frac{y}{r}$$

Note 2. Rule to change a cartesian equation to polar equation

In the given equation, put $x = r\cos\theta$, $y = r\sin\theta$, and simplify the result. The resulting equation is the required equation.

Note 3. Rule to change a polar equation to cartesian equation.

(*i*) In the given equation, put $\cos\theta = \dfrac{x}{r}$, $\sin\theta = \dfrac{y}{r}$, and clear off fractions.

(*ii*) Put $r = \sqrt{x^2 + y^2}$, and express the resulting equation in the rational form (*i.e.*, free from fractional powers). The resulting equation is the required equation.

ILLUSTRATIVE EXAMPLES

Example 1. Find the cartesian coordinates of the points (2, 60°), (– 2, 30°).

Sol. Let (x, y) be cartesian coordinates of (2, 60°) *i.e.* $\left(2, \dfrac{\pi}{3}\right)$.

$$\therefore \quad x = 2\cos\frac{\pi}{3} = 2 \times \frac{1}{2} = 1$$

$$y = 2\sin\frac{\pi}{3} = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

$\therefore$ required cartesian coordinates of (2, 60°) are $(1, \sqrt{3})$.

Again let (x, y) be cartesian coordinates of (– 2, 30°) *i.e.* $\left(-2, \dfrac{\pi}{6}\right)$

$$\therefore \quad x = -2\cos\frac{\pi}{6} = -2 \times \frac{\sqrt{3}}{2} = -\sqrt{3}$$

$$y = -2\sin\frac{\pi}{6} = -2 \times \frac{1}{2} = -1$$

$\therefore$ required cartesian coordinates of (– 2, 30°) are $(-\sqrt{3}, -1)$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 2. Find the cartesian coordinates for the point $\left(-3, -\frac{5\pi}{6}\right)$.

Sol. Let (x, y) be cartesian coordinates of $\left(-3, -\frac{5\pi}{6}\right)$.

$$\therefore \quad x = -3\cos\left(-\frac{5\pi}{6}\right) = -3\cos\frac{5\pi}{6} = -3\cos\left(\pi - \frac{\pi}{6}\right)$$

$$= 3\cos\frac{\pi}{6} = 3 \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

$$y = -3\sin\left(-\frac{5\pi}{6}\right) = 3\sin\frac{5\pi}{6} = 3\sin\left(\pi - \frac{\pi}{6}\right)$$

$$= 3\sin\frac{\pi}{6} = 3 \times \frac{1}{2} = \frac{3}{2}$$

$\therefore$ required cartesian coordinates of $\left(-3, -\frac{5\pi}{6}\right)$ are $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$.

Example 3. Find the polar coordinates for the points with the cartesian coordinates $(-\sqrt{3}, 1)$.

Sol. Let (r, θ) be the point corresponding to $(-\sqrt{3}, 1)$

$$\therefore \quad -\sqrt{3} = r\cos\theta \qquad \text{...(1)}$$

$$\text{and} \quad 1 = r\sin\theta \qquad \text{...(2)}$$

Squaring and adding (1) and (2), we get

$$3 + 1 = r^2(\cos^2\theta + \sin^2\theta) \quad \text{or} \quad 4 = r^2 \Rightarrow r = 2$$

$\therefore$ from (1) and (2), we get

$$\cos\theta = -\frac{\sqrt{3}}{2}, \ \sin\theta = \frac{1}{2}$$

$$\therefore \quad \tan\theta = -\frac{1}{\sqrt{3}} \Rightarrow \theta = \pi - \frac{\pi}{6}$$

[$\because$ θ lies in IInd quadrant]

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

 Gandhi Memorial College Of Education Bantalab Jammu

$\Rightarrow \qquad \theta = \dfrac{5\pi}{6}.$

$\therefore$ polar coordinates of $(-\sqrt{3}, 1)$ are $\left(2, \dfrac{5\pi}{6}\right)$.

Example 4. Find the polar coordinates of the point whose cartesian coordinates are $(-2, 2)$.

Sol. Let (r, θ) be the point corresponding to $(-2, 2)$

$\therefore \quad -2 = r\cos\theta \qquad \ldots(1)$

and $\quad 2 = r\sin\theta \qquad \ldots(2)$

Squaring and adding (1) and (2), we get

$$4+4 = r^2(\cos^2\theta + \sin^2\theta) \quad \text{or } 8 = r^2 \Rightarrow r = 2\sqrt{2}$$

$\therefore$ from (1) and (2), we get

$$\cos\theta = -\frac{1}{\sqrt{2}}, \quad \sin\theta = \frac{1}{\sqrt{2}}$$

$\therefore \quad \tan\theta = -1 \Rightarrow \theta = \pi - \dfrac{\pi}{4}$ [$\because$ θ lies in IInd quadrant]

$\Rightarrow \quad \theta = \dfrac{3\pi}{4}$

$\therefore$ polar coordinates of $(-2, 2)$ are $\left(2\sqrt{2}\cos\dfrac{3\pi}{4}, 2\sqrt{2}\sin\dfrac{3\pi}{4}\right)$

Example 5. Transform $x^2 + y^2 - 2x + 2y = 0$ into polar form.

Sol. The given equation is

$$x^2 + y^2 - 2x + 2y = 0$$

Putting $x = r\cos\theta$, $y = r\sin\theta$ in (1), we get

$$r^2\cos^2\theta + r^2\sin^2\theta - 2r\cos\theta + 2r\sin\theta = 0$$

or $\quad r^2(\cos^2\theta + \sin^2\theta) - 2r\cos\theta + 2r\sin\theta = 0$

or $\quad r^2 - 2r\cos\theta + 2r\sin\theta = 0$

or $\quad r - 2\cos\theta + 2\sin\theta = 0$

or $\quad r = 2(\cos\theta - \sin\theta)$

which is required polar form.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 6. Transform to polar co-ordinates the equation $(x^2+y^2)^2 = a^2(x^2-y^2)$.

Sol. The given equation is $(x^2+y^2)^2 = a^2(x^2-y^2)$ ...(1)

Putting $x = r\cos\theta, y = r\sin\theta$ in (1), we get

$$(r^2\cos^2\theta + r^2\sin^2\theta)^2 = a^2(r^2\cos^2\theta - r^2\sin^2\theta)$$

or $$r^4(\cos^2\theta+\sin^2\theta)^2 = a^2r^2(\cos^2\theta-\sin^2\theta)$$

or $$r^2.(1)^2 = a^2\cos 2\theta$$

$\therefore$ $r^2 = a^2\cos 2\theta$, which is the required equation.

Example 7. Transform the equation $xy^3 + x^3y = a^2$ to polar form. (J.U. 2016)

Sol. The given equation is

$$xy^3 + x^3y = a^2 \quad ...(1)$$

Putting $x = r\cos\theta,\ y = r\sin\theta$ in (1), we get

$$r\cos\theta\,.\,r^3\sin^3\theta + r^3\cos^3\theta\,.\,r\sin\theta = a^2$$

or $$r^4\sin^3\theta\cos\theta + r^4\cos^3\theta\sin\theta = a^2$$

or $$r^4\sin\theta\cos\theta(\sin^2\theta+\cos^2\theta) = a^2$$

or $$r^4\sin\theta\cos\theta\times 1 = a^2$$

or $$r^4(2\sin\theta\cos\theta) = 2a^2$$

or $r^4\sin 2\theta = 2a^2$, which is required equation.

Example 8. Find the equivalent cartesian equation of $r^2 = -6r\sin\theta$. (J.U. 2016)

Sol. The given equation is

$$r^2 = -6r\sin\theta \quad ...(1)$$

Putting $\sin\theta = \dfrac{y}{r}$ in (1), we get

$$r^2 = -6r\times\frac{y}{r} \quad \text{or} \quad r^2 = -6y$$

Putting $r^2 = x^2+y^2$, we get

$$x^2+y^2 = -6y \quad \text{or} \quad x^2+y^2+6y = 0$$

which is required equation.

Example 9. Change to cartesian co-ordinates the equation :

$$r^{\frac{1}{2}} = a^{\frac{1}{2}}\cos\frac{\theta}{2}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Sol. The given equation is

$$r^{\frac{1}{2}} = a^{\frac{1}{2}} \cos\frac{\theta}{2} \qquad \text{or} \qquad r = a\cos^2\frac{\theta}{2}$$

$$\text{or} \quad r = a\left(\frac{1+\cos\theta}{2}\right) \qquad \text{or} \qquad 2r = a + a\cos\theta$$

$$\text{or} \quad 2r = a + \frac{ax}{r} \qquad \left[\because \cos\theta = \frac{x}{r}\right]$$

$$\text{or} \quad 2r^2 = ar + ax$$

$$\text{or} \quad 2x^2 + 2y^2 = ar + ax \qquad [\because r^2 = x^2 + y^2]$$

$$\text{or} \quad 2x^2 + 2y^2 - ax = ar \qquad \text{or} \qquad (2x^2 + 2y^2 - ax)^2 = a^2r^2$$

$$\text{or} \quad (2x^2 + 2y^2 - ax)^2 = a^2(x^2 + y^2)$$

which is required equation.

Example 10. Find the cartesian equation for the curve

$$r = \frac{a}{\theta}, \text{ where } a > 0.$$

Sol. The given equation is

$$r = \frac{a}{\theta} \quad \text{or} \quad r\theta = a, \quad a > 0 \qquad \ldots(1)$$

$$\therefore \quad \sqrt{x^2 + y^2}\,\tan^{-1}\left(\frac{y}{x}\right) = a \quad \text{or} \quad \tan^{-1}\left(\frac{y}{x}\right) = \frac{a}{\sqrt{x^2 + y^2}}$$

which is required equation.

Example 11. Show that $\left(\frac{1}{2}, \frac{3\pi}{2}\right)$ lies on $r = \sin\left(\frac{\theta}{3}\right)$.

Sol. The equation of curve is $r = -\sin\left(\frac{\theta}{3}\right)$

The point $\left(\frac{1}{2}, \frac{3\pi}{2}\right)$ will be on this curve

if $\quad \frac{1}{2} = -\sin\left(\frac{3\pi}{2\times 3}\right)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

i.e. if $\frac{1}{2} = -\sin\frac{\pi}{2}$

i.e. if $\frac{1}{2} = -1$, which is not true

$\therefore$ point $\left(\frac{1}{2}, \frac{3\pi}{2}\right)$ does not lie on the curve $r = -\sin\left(\frac{\theta}{3}\right)$.

EXERCISE 12 (*a*)

1. (*a*) Plot the following points whose polar co-ordinates are

(*i*) $\left(3, \frac{\pi}{4}\right)$ (*ii*) $\left(-2, \frac{\pi}{3}\right)$ (*iii*) $\left(3, -\frac{\pi}{4}\right)$

(*iv*) $\left(-3, -\frac{\pi}{4}\right)$ (*v*) $\left(2, \frac{8\pi}{3}\right)$

Also determine their corresponding Cartesian co-ordinates.

(*b*) Find the cartesian coordinates of the points (2, 180°), $\left(3, \frac{\pi}{2}\right)$.

(*c*) Find the cartesian coordinates of the point P whose polar coordinates are $\left(6, \frac{\pi}{6}\right)$.

2. (*a*) Determine the polar co-ordinates of point $\left(2, 2\sqrt{3}\right)$.

(*b*) Find the polar coordinates of the point P whose co-ordinates are $(-5, 0)$, $r \geq 0$ and $0 \leq \theta \leq 2\pi$.

3. Determine the polar co-ordinates for each of the following points in rectangular co-ordinates :

(*i*) $(1, -1)$ (*ii*) $\left(2, -2\sqrt{3}\right)$ (*iii*) $(-5, -12)$

4. Determine the polar co-ordinates for the following points given in rectangular co-ordinates :

(*i*) $(2, 1)$ (*ii*) $(-1, 1)$ (*iii*) $(-2, 1)$

(*iv*) $(-3, 4)$ (*v*) $(2, -2)$ (*vi*) $(-1, -1)$

(*vii*) $(-2, -1)$ (*viii*) $(-7, -12)$ (*ix*) $(-3, -4)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

5. Change to polar co-ordinates the equation $x^2 = y^2 (2a - x)$.

6. Transform $x^2 + y^2 - 2x + 2y = 0$ into polar form.

7. Transform to polar co-ordinates the equation :

(*i*) $y = x \tan \alpha$ (*ii*) $x^2 + y^2 = a^2$

(*iii*) $x^2 - y^2 = a^2$

(*iv*) $x^2 + y^2 = 2ax$ (*v*) $x^3 = y^2 (2a - x)$

(*vi*) $x^2 - y^2 = 2ay$ (*vii*) $(x^2 + y^2)^2 = 4a^2 x^2 y^2$

8. Change to cartesian co-ordinates the equation $r = a \sin 2\theta$.

9. Find the cartesian transform of the equations

(*i*) $r = \cos \theta$

(*ii*) $r^2 = \sin 2\theta$

10. Find the cartesian transform of the equations :

(*i*) $r = \sin \theta$ (*ii*) $r^2 = \cos 2\theta$

11. Change to cartesian co-ordinates the equation :

(*i*) $r = a$ (*ii*) $\theta = \tan^{-1} m$ (*iii*) $r = 2a \cos \theta$

(*iv*) $r^2 \cos 2\theta = a^2$ (*v*) $r^2 \sin 2\theta = 2c^2$ (*vi*) $r^2 = a^2 \cos 2\theta$

(*vii*) $r^{\frac{1}{2}} \cos \frac{\theta}{2} = a^{\frac{1}{2}}$ (*viii*) $r^{\frac{1}{2}} = a^{\frac{1}{2}} \sin \frac{\theta}{2}$

(*ix*) $r(\cos 3\theta + \sin 3\theta) = 5k \sin \theta \cos \theta$

12. Transform to rectangular Cartesian co-ordinates the equations :

(*i*) $r \cos(\theta - \alpha) = p$ (*ii*) $r \sin \theta = -1$ (*iii*) $r = 2 \sec \theta \tan \theta$.

ANSWERS

1.(*a*) (*i*) $\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ (*ii*) $\left(-1, -\sqrt{3}\right)$ (*iii*) $\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$

(*iv*) $\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ (*v*) $\left(-1, \sqrt{3}\right)$

(*b*) $(-2, 0)$; $(0, 3)$ (*c*) $\left(3\sqrt{3}, 3\right)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Education Bantalab Jammu

2. (*a*) $\left(4, \frac{\pi}{3}\right)$ (*b*) $(5, \pi)$

3. (*i*) $\left(\sqrt{2}, -\frac{\pi}{4}\right)$ (*ii*) $\left(4, -\frac{\pi}{3}\right)$ (*iii*) $\left(13, \tan^{-1}\frac{12}{5}\right)$

4. (*i*) $\left(\sqrt{5}, \tan^{-1}\left(\frac{1}{2}\right)\right)$ (*ii*) $\left(\sqrt{2}, \frac{3\pi}{4}\right)$ (*iii*) $\left(\sqrt{5}, \tan^{-1}\left(-\frac{1}{2}\right)\right)$

(*iv*) $\left(5, \tan^{-1}\left(-\frac{4}{3}\right)\right)$ (*v*) $\left(2\sqrt{2}, -\frac{\pi}{4}\right)$ (*vi*) $\left(\sqrt{2}, \frac{5\pi}{4}\right)$

(*vii*) $\left(\sqrt{5}, \tan^{-1}\left(\frac{1}{2}\right)\right)$ (*viii*) $\left(\sqrt{193}, \tan^{-1}\frac{12}{7}\right)$ (*ix*) $\left(5, \tan^{-1}\frac{4}{3}\right)$

5. $\cos^2\theta = \sin^2\theta\,(2a - r\cos\theta)$

6. $2\cos\theta - 2\sin\theta = r$

7. (*i*) $\theta = \alpha$ (*ii*) $r^2 = a^2$ (*iii*) $r^2\cos 2\theta = a^2$

(*iv*) $r = 2a\cos\theta$ (*v*) $r\cos\theta = 2a\sin^2\theta$ (*vi*) $r\cos 2\theta = 2a\sin\theta$

(*vii*) $1 = a^2\sin^2 2\theta$

8. $(x^2+y^2)^3 = 4a^2x^2y^2$

9. (*i*) $x^2+y^2-x=0$ (*ii*) $(x^2+y^2)^2 = 2xy$

10. (*i*) $x^2+y^2-y=0$ (*ii*) $(x^2+y^2)^2 = x^2-y^2$

11. (*i*) $x^2+y^2 = a^2$ (*ii*) $y = mx$ (*iii*) $x^2+y^2 = 2ax$

(*iv*) $x^2-y^2 = a^2$ (*v*) $xy = c^2$

(*vi*) $(x^2+y^2)^2 = a^2(x^2-y^2)$ (*vii*) $y^2 = 4a(a-x)$

(*viii*) $(2x^2+2y^2+ax)^2 = a^2(x^2+y^2)$

(*ix*) $x^3-3xy^2+3x^2y-y^3 = 5kxy$

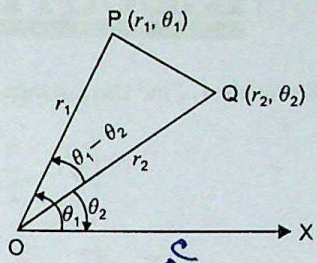
12. (*i*) $x\cos\alpha + y\sin\alpha = p$ (*ii*) $y+1=0$ (*iii*) $x^2 = 2y$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Art-3. Distance between two points P (r_1, θ_1) and Q (r_2, θ_2)

Let P (r_1, θ_1) and Q (r_2, θ_2) be two given points.



Join OP, OQ. Then

$$|OP| = r_1, \quad |OQ| = r_2$$

$$\angle XOP = \theta_1, \quad \angle XOQ = \theta_2$$

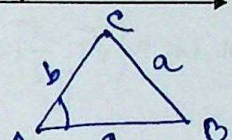
$$\therefore \angle QOP = \theta_1 - \theta_2$$

In ΔQOP, by cosine formula

$$|PQ|^2 = |OP|^2 + |OQ|^2 - 2\,|OP|\,|OQ| \cos \angle QOP$$

$$= r_1^2 + r_2^2 - 2\, r_1 r_2 \cos(\theta_1 - \theta_2)$$

$$\therefore \quad |PQ| = \sqrt{r_1^2 + r_2^2 - 2 r_1 r_2 \cos(\theta_1 - \theta_2)}.$$

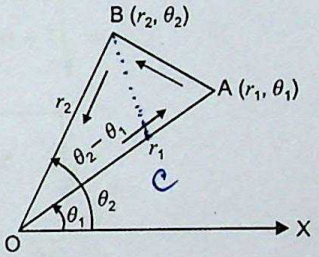


$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Art-4 (*a*). Area of triangle with one vertex at origin

Let OAB be the given triangle with one vertex at O and the triangle described in the counter clockwise direction, where

A (r_1, θ_1) and B (r_2, θ_2)



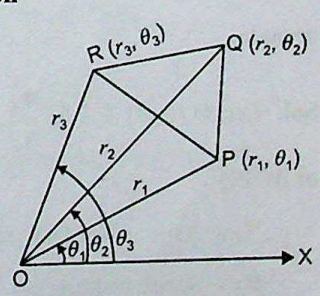
$\therefore$ area of the ΔOAB

$$= \frac{1}{2}\, |OA|\,|OB| \sin(\theta_2 - \theta_1)$$

$$= \frac{1}{2}\, r_1 r_2 \sin(\theta_2 - \theta_1)$$

Art-4 (*b*). Area of a triangle whose vertices are given

Let P (r_1, θ_1), Q (r_2, θ_2), R (r_3, θ_3) be the vertices of ΔPQR, then



$$|OP| = r_1, \quad |OQ| = r_2,$$

$$|OR| = r_3$$

$$\angle XOP = \theta_1, \quad \angle XOQ = \theta_2,$$

$$\angle XOR = \theta_3$$

$$\therefore \angle POQ = \theta_2 - \theta_1; \ \angle QOR = \theta_3 - \theta_2$$

$$\angle POR = \theta_3 - \theta_1$$

Now area of ΔPQR

= area of ΔPOQ + area of ΔQOR – area of ΔPOR

$$= \frac{1}{2} r_1 r_2 \sin(\theta_2 - \theta_1) + \frac{1}{2} r_2 r_3 \sin(\theta_3 - \theta_2) - \frac{1}{2} r_1 r_3 \sin(\theta_3 - \theta_1)$$

Hence

$$\Delta PQR = \frac{1}{2}\, [r_1 r_2 \sin(\theta_2 - \theta_1) + r_2 r_3 \sin(\theta_3 - \theta_2) - r_1 r_3 \sin(\theta_3 - \theta_1)].$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

ILLUSTRATIVE EXAMPLES

Example 1. Find the distance between the points

$$\left(\sqrt{7},\ \frac{\pi}{2}\right) \text{ and } \left(3\sqrt{7},\ \frac{\pi}{6}\right).$$

Sol. Let $P\left(\sqrt{7},\ \frac{\pi}{2}\right)$ and $Q\left(3\sqrt{7},\ \frac{\pi}{6}\right)$ be the given points.

$$\therefore \quad |PQ| = \sqrt{\left(\sqrt{7}\right)^2 + \left(3\sqrt{7}\right)^2 - 2\left(\sqrt{7}\right)\left(3\sqrt{7}\right)\cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right)}$$

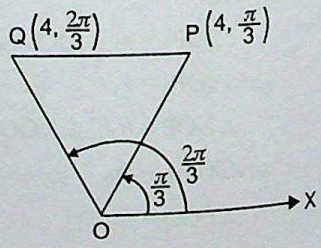
$$= \sqrt{7 + 63 - 42\cos\frac{\pi}{3}}$$

$$= \sqrt{7 + 63 - 42\left(\frac{1}{2}\right)} = \sqrt{7 + 63 - 21}$$

$$= \sqrt{49} = 7$$

Example 2. Show that the points (0, 0), $\left(4, \frac{\pi}{3}\right)$ and $\left(4, \frac{2\pi}{3}\right)$ form an equilateral triangle.

Sol. Let O (0, 0), $P\left(4, \frac{\pi}{3}\right)$, $Q\left(4, \frac{2\pi}{3}\right)$ be the vertices of ΔOPQ



Now $|OP| = 4$, $|OQ| = 4$

By distance formula

$$|PQ| = \sqrt{16 + 16 - 2(4)(4)\cos\left(\frac{2\pi}{3} - \frac{\pi}{3}\right)}$$

$$= \sqrt{32 - 32\cos\frac{\pi}{3}} = \sqrt{32 - 16} = 4$$

$$\therefore \quad |OP| = |OQ| = |PQ|$$

Hence ΔOPQ is an equilateral triangle.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 3. Find the area of the triangle whose vertices are $\left(4, \frac{\pi}{6}\right), \left(4, \frac{\pi}{2}\right), \left(8, \frac{2\pi}{3}\right)$.

Sol. Let $P\left(4, \frac{\pi}{6}\right)$, $Q\left(4, \frac{\pi}{2}\right)$, $R\left(8, \frac{2\pi}{3}\right)$ be the vertices of ΔPQR

Here $r_1 = 4,\ \theta_1 = \frac{\pi}{6},\ r_2 = 4,\ \theta_2 = \frac{\pi}{2},\ r_3 = 8,\ \theta_3 = \frac{2\pi}{3}$

Using,

$$\Delta PQR = \frac{1}{2}\left[r_1 r_2 \sin(\theta_2 - \theta_1) + r_2 r_3 \sin(\theta_3 - \theta_2) - r_1 r_3 \sin(\theta_3 - \theta_1)\right]$$

we get,

$$\Delta PQR = \frac{1}{2}\left[16 \sin\left(\frac{\pi}{2} - \frac{\pi}{6}\right) + 32 \sin\left(\frac{2\pi}{3} - \frac{\pi}{2}\right) - 32 \sin\left(\frac{2\pi}{3} - \frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}\left[16 \sin\frac{\pi}{3} + 32 \sin\frac{\pi}{6} - 32 \sin\frac{\pi}{2}\right]$$

$$= \frac{1}{2}\left[\frac{16\sqrt{3}}{2} + \frac{32}{2} - 32\right] = \frac{1}{2}\left[8\sqrt{3} - 16\right]$$

$$= 8 - 4\sqrt{3} \text{ (in magnitude).}$$

EXERCISE 12 (*b*)

1. Find the distance between the points whose polar co-ordinates are

   (*i*) $\left(3, \frac{\pi}{4}\right)$ and $\left(2, \frac{\pi}{3}\right)$ (*ii*) $\left(-3, \frac{\pi}{4}\right)$ and $\left(4, \frac{7\pi}{12}\right)$.

2. Find the area of the triangle, polar co-ordinates of whose angular points are

   $(a, \theta), \left(2a, \theta + \frac{\pi}{3}\right), \left(3a, \theta + \frac{2\pi}{3}\right)$

ANSWERS

1. (*i*) $\sqrt{13 - 3\sqrt{2}\left(1 + \sqrt{3}\right)}$ (*ii*)

2. $\frac{5\sqrt{3}a^2}{4}$ sq. units.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

13

GRAPHING TECHNIQUES IN POLAR CO-ORDINATES

Art-1. Procedure for Tracing Polar Curves

We shall keep in mind the following points for tracing the graphs of the equation $f(r, \theta) = 0$.

I. Symmetry.

(*i*) *Symmetry about the initial line or x-axis* : If the equation of the curve remains unchanged when θ is changed to $-\theta$, the curve is symmetrical about the initial line.

(*ii*) *Symmetry about the line* $\theta = \frac{\pi}{2}$ *or y-axis* : If the equation of the curve remains unchanged when θ is changed to $\pi - \theta$ or when θ is changed to $-\theta$ and r to $-r$, the curve is symmetrical about the line $\theta = \frac{\pi}{2}$.

(*iii*) *Symmetry about the line* $\theta = \frac{\pi}{4}$ *or* $y = x$: If the equation of the curve remains unchanged when θ is changed to $\frac{\pi}{2} - \theta$, the curve is said to be symmetrical about the line $\theta = \frac{\pi}{4}$.

(*iv*) *Symmetrical about the lie* $\theta = \frac{3\pi}{4}$ or $y = -x$: if the equation of the curve remains unchanged when θ is changed to $\frac{3\pi}{2} - \theta$, the curve is said to be symmetrical about the line $\theta = \frac{3\pi}{4}$.

(*v*) *Symmetry about the pole* : If the equation of the curve remains unchanged when r is changed to $-r$, the curve is said to be symmetrical about the pole.

II. Pole

(*i*) Find whether the curve passes through the pole or not. It can be done by putting $r = 0$ in the equation and then finding some real value of θ. If it is not possible to find a real value of θ for which $r = 0$, then the curve does not pass through the pole.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

(*ii*) Find the tangents at the pole. Putting $r = 0$, the real values of θ give the tangents at the pole.

(*iii*) Find the points where the curve meets the initial line and the line $\theta = \frac{\pi}{2}$. , $-\pi/2$

III. Value of ϕ

Find ϕ from the result $\tan\phi = r\dfrac{d\theta}{dr}$. Then find the points where $\phi = 0$ or $\dfrac{\pi}{2}$.

IV. Asymptotes

If $r \to \infty$ as $\theta \to \theta_1$ (any fixed number), then there is an asymptote. Find it by the method given below :

(*i*) Write down the given equation as $\dfrac{1}{r} = f(\theta)$, say.

(*ii*) Equate $f(\theta)$ to zero and solve for θ. Let the roots be $\theta_1, \theta_2, \ldots\ldots$

(*iii*) Find $f'(\theta)$ and calculate it at $\theta = \theta_1, \theta_2, \ldots\ldots$

(*iv*) Asymptotes are $r\sin(\theta - \theta_1) = \dfrac{1}{f'(\theta)}, r\sin(\theta - \theta_2) = \dfrac{1}{f'(\theta_2)}, \ldots$

V. Special Points

Find some points on the curve for convenient values of θ.

VI. Region

Solve the given equation for r or θ. Find the region in which the curve does not lie. This can be done in the following manner.

(*i*) No part of the curve lies between $\theta = \alpha$ and $\theta = \beta$ if for $\alpha < \theta < \beta$, r is imaginary.

(*ii*) If the greatest numerical value of r be a, the curve lies entirely within the circle $r = a$. If the least numerical value of r be b, the curve lies outside the circle $r = b$.

ILLUSTRATIVE EXAMPLES

Example 1. (*a*) Trace the curve $r = a(1 + \cos\theta)$, $a > 0$.

(*b*) Trace the curve $r = 5(1 + \cos\theta)$

Sol. (*a*) The equation of the curve is $r = a(1 + \cos\theta)$ ...(1)

I. Symmetry : The equation of the curve remains unchanged when θ is changed to $-\theta$.

$\therefore$ curve is symmetrical about the initial line.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

II. Pole : Putting $r = 0$ in (1), we get

$$a(1 + \cos\theta) = 0 \text{ or } \cos\theta = -1$$

$\therefore \quad \theta = \pi$

$\therefore$ pole lies on the curve and tangent at the pole is $\theta = \pi$.

The curve cuts the initial line $\theta = 0$ at $(2a, 0)$ and the lines $\theta = \pm\dfrac{\pi}{2}$ at

$$\left(a, \frac{\pi}{2}\right), \left(a, -\frac{\pi}{2}\right).$$

III. Value of ϕ

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\therefore \quad \tan\phi = r\frac{d\theta}{dr} = a(1+\cos\theta)\times\frac{1}{-a\sin\theta} = -\frac{2\cos^2\dfrac{\theta}{2}}{2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}$$

$$\therefore \quad \tan\phi = -\cot\frac{\theta}{2} \quad \Rightarrow \quad \tan\phi = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right) \quad \Rightarrow \quad \phi = \frac{\pi}{2}+\frac{\theta}{2}$$

$\therefore \quad \phi = \dfrac{\pi}{2}$ when $\theta = 0$, $r = 2a$

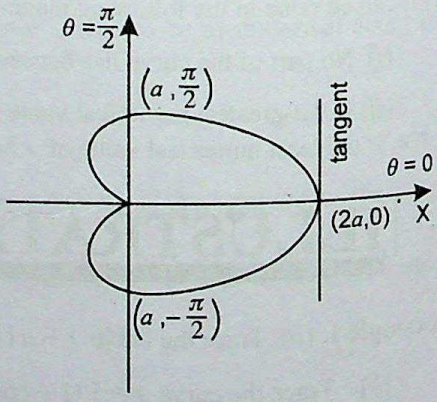
$\therefore$ at $(2a, 0)$, the tangent is perpendicular to initial line.

IV. Asymptotes : Since r does not tend to infinity for any finite value of θ.

$\therefore$ curve has got no asymptote

V. Special Points. We have

θ :	0	$\dfrac{\pi}{4}$	$\dfrac{\pi}{2}$	π
r :	$2a$	$a\left(1+\dfrac{1}{\sqrt{2}}\right)$	a	0



VI. Region. Since $r = a(1 + \cos\theta)$

$\therefore$ max. value of $r = 2a$

$\therefore$ curve lies entirely within the circle $r = 2a$

When θ increases from 0 to π, r remains positive and decreases from $2a$ to 0.

When θ increases from π to 2π, r remains positive and increases from 0 to $2a$.

The shape of the curve is as shown in the figure.

(*b*) Take $a = 5$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 2. Trace the curve $r = a + b\cos\theta,\ a > b.$

Sol. The equation of curve is $r = a + b\cos\theta$ ...(1)

I. Symmetry. The equation of the curve remains unchanged when θ is changed to $-\theta$

$\therefore$ curve is symmetrical about the initial line.

II. Pole. Putting $r = 0$ in (1), we get,

$$a + b\cos\theta = 0 \text{ or } \cos\theta = -\frac{a}{b}$$

$$\therefore \quad |\cos\theta| = \frac{a}{b} > 1 \text{ as } a > b$$

This is not possible

$\therefore$ for no value of θ, r is equal to zero

$\therefore$ curve does not pass through the pole.

III. Value of ϕ

$$\frac{dr}{d\theta} = -b\sin\theta$$

$$\therefore \quad \tan\phi = r\frac{d\theta}{dr} = \frac{a + b\cos\theta}{-b\sin\theta} \neq 0$$

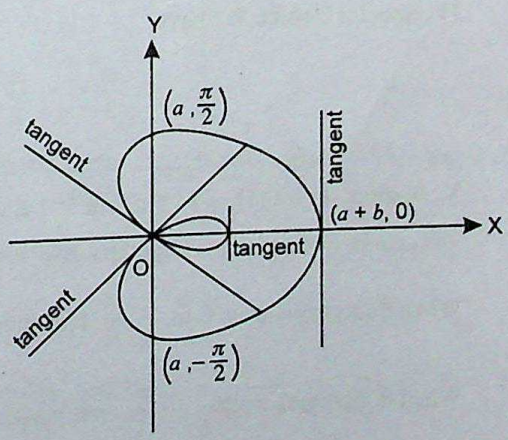
$\therefore$ $\phi \neq 0$ at any point

When $\theta = 0$, $r = a + b$, then $\phi = \frac{\pi}{2}$

$\therefore$ at $(a + b, 0)$, the tangent is perpendicular to initial line.

IV. Asymptotes. Since r does not tend to infinity for any finite value of θ

$\therefore$ curve has got no asymptote.



V. Special Points. We have

θ:	0	$\frac{\pi}{2}$	π
r:	$a + b$	a	$a - b$

VI. Region. Since $r = a + b\cos\theta$ and $|\cos\theta| \leq 1$

$\therefore$ $|r| \leq a + b$

$\therefore$ curve lies entirely within the circle $r = a + b$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

When θ increases from 0 to $\frac{\pi}{2}$, r remains positive but decreases from $a+b$ to a.

When θ increase from $\frac{\pi}{2}$ to π, r decreases from a to $a-b$.

The shape of the curve is as shown in the figure.

Example 3. Trace the curve $r = a \sin 3\theta$, $a > 0$. (J.U. 2016)

Sol. The equation of the curve is $r = a \sin 3\theta$ ...(1)

I. Symmetry. The equation of the curve remains unchanged when θ is changed to $\pi - \theta$.

$\therefore$ curve is symmetrical about the line $\theta = \frac{\pi}{2}$.

II. Pole. Putting $r = 0$ in (1), we get

$a \sin 3\theta = 0$ or $\sin 3\theta = 0$

$\therefore$ $3\theta = 0, \pi, \quad 2\pi, \quad 3\pi, \ldots\ldots$

or $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \quad \pi, \ldots\ldots$

$\therefore$ the curve passes through the pole and the tangents at pole are

$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$ as the other values of θ give the same tangents.

III. Asymptotes. Since r does not tend to infinity for any finite value of θ.

$\therefore$ curve has get no asymptote.

IV. Special Points. We have

θ:	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r:	0	a	0	$-a$	0	a	0

V. Region. From (1), $|r| = a|\sin 3\theta| \le a$

$\therefore$ curve lies entirely within the circle $r = a$.

When θ increases from 0 to $\frac{\pi}{6}$, r is positive and increases from 0 to a.

When θ increases from $\frac{\pi}{6}$ to $\frac{\pi}{3}$, r is positive and decreases from a to 0.

Thus we get a loop between the liens $\theta = 0$ and $\theta = \frac{\pi}{3}$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

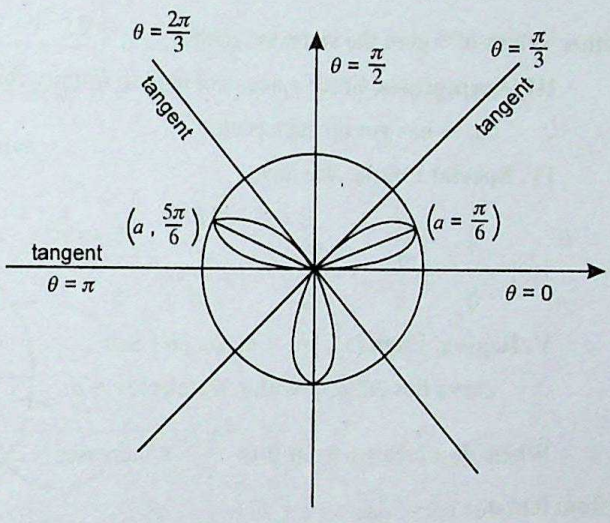
Gandhi Memorial College Of Education Bantalab Jammu

As θ increases from $\frac{\pi}{3}$ to $\frac{\pi}{2}$, r is negative and numerically increases from 0 to a and when θ increases from $\frac{\pi}{2}$ to $\frac{2\pi}{3}$, r is negative and numerically decreases from a to 0.

$\therefore$ we get another loop between the lies $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$.

When θ increases from $\frac{2\pi}{3}$ to $\frac{5\pi}{6}$, r is positive and increases from 0 to a and when θ increases from $\frac{5\pi}{6}$ to π, r is positive and decreases from a to 0.

$\therefore$ we get third loop between the lines $\theta = \frac{2\pi}{3}$ and $\theta = \pi$.



When θ varies from π to 2π, the same loops are repeated and we do not get any new loop as r is periodic.

The curve $r = a \sin n\theta$ or $r = a \cos n\theta$

consists of n or $2n$ loops according as n is odd or even.

Example 4. Trace the curve $r = a \sin 2\theta$.

Sol. The equation of given curve is

$$r = a \sin 2\theta \qquad \ldots(1)$$

I. Symmetry. The equation of the curve remains unchanged when θ is changed to $\frac{\pi}{2} - \theta$.

$\therefore$ curve is symmetrical about the line $\theta = \frac{\pi}{4}$.

Also the equation of the curve remains unchanged when θ is changed to $-\theta$ and r to $-r$.

$\therefore$ curve is symmetrical about the line $\theta = \frac{\pi}{2}$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

II. Pole. Putting $r = 0$ in (1), we get,

$$a \sin 2\theta = 0 \text{ or } \sin 2\theta = 0$$

$$\therefore \quad \theta = 0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}, \ldots\ldots$$

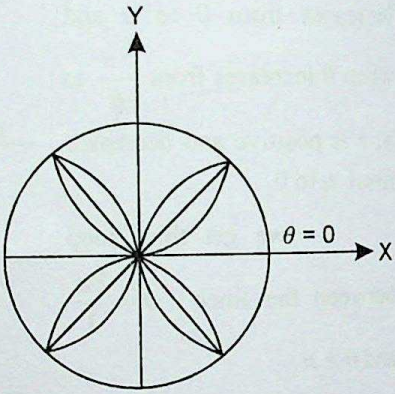
Thus the curve passes through the pole and tangents at the pole are $\theta = 0, \theta = \frac{\pi}{2}$ as other values of θ give the same tangents.

III. Asymptotes. Since r does not tend to infinity for any finite value of θ.

$\therefore$ curve has got no asymptote.

IV. Special Points. We have

θ:	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r:	0	a	0	$-a$	0



V. Region. From (1), $|r| = a|\sin 2\theta| \le a$

$\therefore$ curve lies entirely within the circle $r = a$.

When θ increases from 0 to $\frac{\pi}{4}$, r increases from 0 to a.

When θ increases from $\frac{\pi}{4}$ to $\frac{\pi}{2}$, r decreases from a to 0.

Thus we get a loop between the lines $\theta = 0$ and $\theta = \frac{\pi}{2}$.

Similarly, we get three other loops.

A rough sketch of the curve is given in the figure.

Example 5. Trace the curve $r = \dfrac{a\theta^2}{1+\theta^2}$

Sol. The equation of curve is

$$r = \frac{a\theta^2}{1+\theta^2} \qquad \ldots(1)$$

I. Symmetry.

The equation of the curve remains uncharged when θ is changed to $-\theta$.

$\therefore$ curve is symmetrical about the initial line.

II. Pole. Putting $r = 0$ in (1), we get, $a\theta^2 = 0$ or $\theta = 0$

$\therefore$ pole lies on the curve and tangent at the pole is $\theta = 0$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

III. Asymptotes.

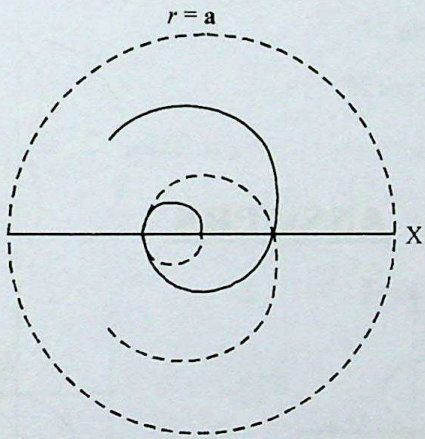
$$\underset{\theta \to \infty}{\text{Lt}}\ r = \underset{\theta \to \infty}{\text{Lt}}\ \frac{a\theta^2}{1+\theta^2} = \underset{\theta \to \infty}{\text{Lt}}\ \frac{a}{\frac{1}{\theta^2}+1} = a$$

$\therefore$ $r = a$ is a circular asymptote.

IV. Region.

$$\frac{dr}{d\theta} = a\left[\frac{(1+\theta^2).2\theta - \theta^2.2\theta}{(1+\theta^2)^2}\right] = \frac{2a\theta}{(1+\theta^2)^2}$$

$\therefore$ for $\theta > 0$, $\frac{dr}{d\theta}$ is positive.



$\therefore$ r constantly increases as θ increases.

Thus r starting from its initial value 0, constantly increases and approaches a as $\theta \to \infty$ so that the point (r, θ) approaches nearer and nearer the circle $r = a$.

A rough sketch of the curve is given in the figure.

Note. Students should be clear in their mind that if the equation of curve is given in polar co-ordinates, it is not a must to do it by the method of polar co-ordinates. Sometimes it becomes very easy to trace a curve after transforming its equation to cartesian co-ordinates.

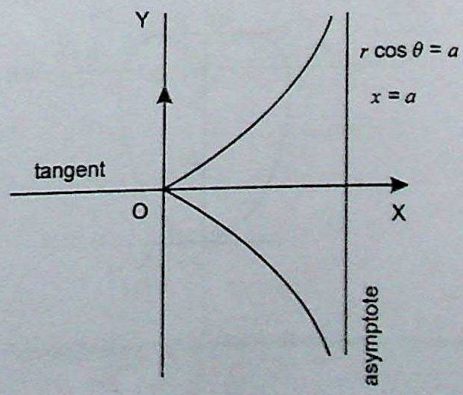
For Example :

Consider the curve $r = \frac{a\sin^2\theta}{\cos\theta}$

This equation can be written as

$$r^2 = a\frac{r^2\sin^2\theta}{r\cos\theta}$$

or $x^2 + y^2 = \frac{ay^2}{x}$



CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

or $x(x^2 + y^2) = a y^2$

or $y^2 (a - x) = x^3$, which can be easily traced.

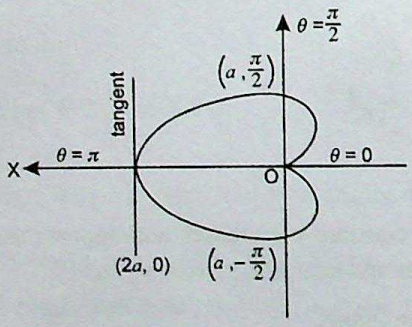
A rough sketch of the curve is given in the figure.

EXERCISE 13 (*a*)

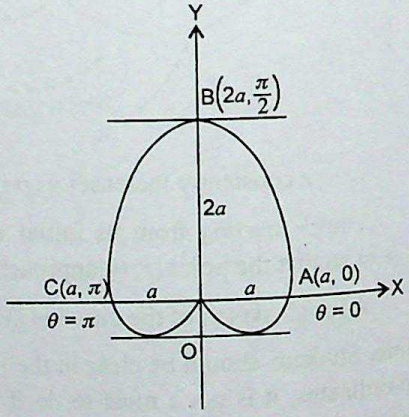
1. Trace the curve $r = a(1 - \cos\theta)$. (J.U. 2016)
2. Trace the curve $r = a(1 + \sin\theta)$
3. Trace the curve $r = a(1 - \sin\theta)$.
4. Trace the curve $r = 2 + 3\cos\theta$
5. Trace the curve $r = a\cos 3\theta$
6. Trace the curve $r = a\cos 2\theta$
7. Trace the curve $r^2 = a^2 \cos 2\theta$
8. Trace the curve $r\theta = a$.

ANSWERS

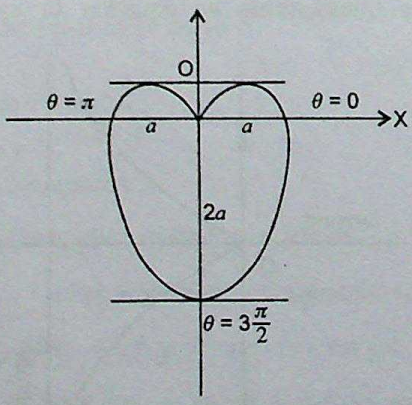
1.



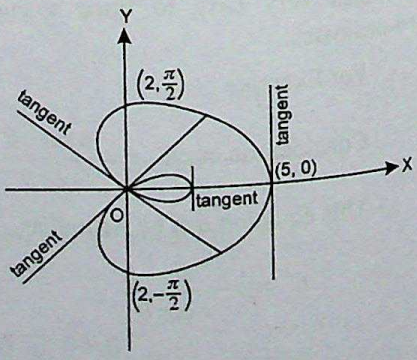
2.



3.



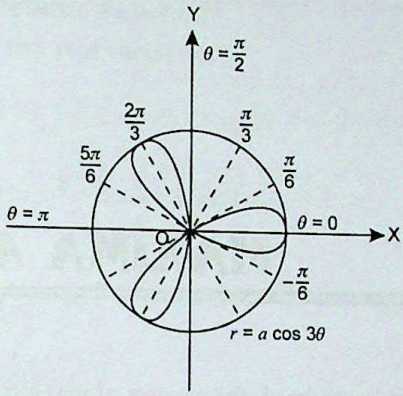
4.



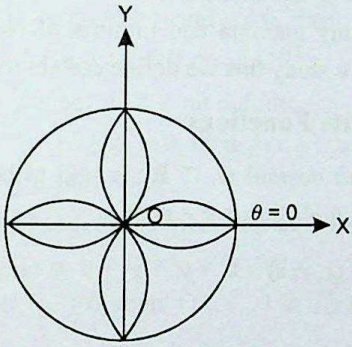
CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Education Bantalab Jammu

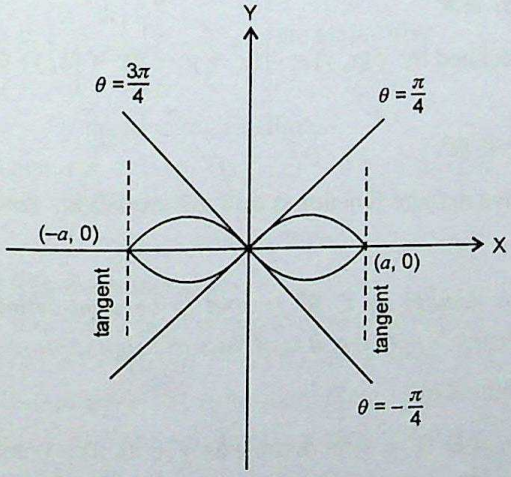
5.



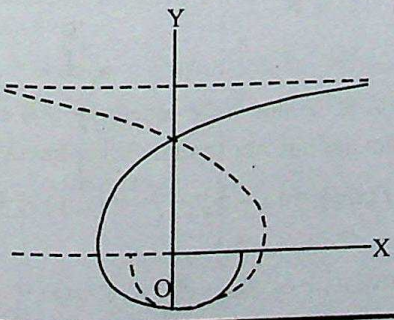
6.



7.



8.



CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

14

MAXIMA AND MINIMA

Art-1. Introduction

We have studied at lower level, the concept of maxima and minima of a real valued function of a single variable. Now we shall study maxima and minima of real valued function of two or more independent variables. To study this we define certain terms as :

Art-2. Definite, Semi-Definite and Indefinite Functions

Definite function : *A real valued function f with domain $D_f \subset \mathbf{R}^n$* is said to be *positive definite if $f(x) > 0$ $\forall x \in D_f$ and negative definite if $f(x) < 0$ $\forall x \in D_f$*

e.g., (*i*) the function $f: \mathbf{R}^3 \to \mathbf{R}$ defined by $f(x, y, z) = x^2 + y^2 + z^2 + 9$ $\forall (x, y, z) \in \mathbf{R}^3$ is positive definite.

Here $f(x, y, z) \geq 9 > 0$ $\forall (x, y, z) \in \boldsymbol{R}^3$

(*ii*) The function $f: \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x, y) = -(x^2 + y^2 + 9)$ $\forall (x, y) \in \mathbf{R}^2$ is negative definite.

Here $f(x, y) \leq -9 < 0$ $\forall (x, y) \in \mathbf{R}^2$.

Note : A positive definite or a negative definite function is said to be a definite function.

Semi-definite functions.

A real valued function f with domain $D_f \subset \boldsymbol{R}^n$ is said to be semi-definite if it vanishes at some points of D_f and when it is not zero, it is of the same sign throughout

e.g., the function $f: \mathbf{R}^3 \to \mathbf{R}$ defined by

$f(x, y, z) = x^2 + y^2 + z^2$ $\forall$ $(x, y, z) \in \mathbf{R}^3$ is semi definite as $f(0, 0, 0) = 0$ and when $(x, y, z) \neq (0, 0, 0)$ then $f(x, y, z) > 0$ where $(x, y, z) \in \mathbf{R}^3$.

Indefinite function

A real valued function f with domain $D_f \subset \mathbf{R}^n$ is said to be indefinite if it can take values which have different signs *i.e.*, it is neither definite nor semi definite function.

e.g., the function $f: \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x, y) = 4x - 5y + 7$ $\forall (x, y) \in \mathbf{R}^2$ is an indefinite function.

Here $f(x, y)$ can be positive, zero or negative.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Art-3. Conditions for a Definite Function

(*i*) Quadratic expression of two real variables.

Let $f(x, y) = a x^2 + 2 h x y + b y^2$, $a \neq 0 = \frac{1}{a}[a^2 x^2 + 2 a h x y + a b y^2]$

$$= \frac{1}{a}\left[(a x + h y)^2 + (a b - h^2) y^2\right] \quad \ldots(1)$$

The following cases arise :

Case I. If $a b - h^2 \geq 0$ and $a > 0$, then

$f(x, y) \geq 0 \; \forall \; (x, y) \in R^2$

$\therefore$ f is positive semi-definite.

Case II. If $a b - h^2 \geq 0$ and $a < 0$, then

$f(x, y) \leq 0 \; \forall \; (x, y) \in R^2$

$\therefore$ f is negative semi definite

Case III. If $a b - h^2 < 0$, then $f(x, y)$ can be of any sign.

$\therefore$ f is indefinite.

(*ii*) **Quadratic expression of three real variables**

Let $f(x, y, z) = a x^2 + b y^2 + c z^2 + 2 f y z + 2 g z x + 2 h x y$

When $a \neq 0$, then $f(x, y, z) \geq 0 \; \forall \; (x, y, z) \in \mathbf{R}^3$

if a, $\begin{vmatrix} a & h \\ h & b \end{vmatrix}$, $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ are all positive

$\therefore$ f is positive semi-definite.

When $a \neq 0$, then $f(x, y, z) \leq 0 \; \forall \; (x, y, z) \in \mathbf{R}^3$ if the above three expressions are alternately negative and positive.

$\therefore$ *f is negative semi-definite.*

Art-4. Maximum Value

A function $f(x, y)$ is said to have a maximum value at $x = a, \; y = b$.

If $f(a, b) > f(a + h, b + k)$ for small values of h and k, positive or negative.

Minimum Value : A function $f(x, y)$ is said to have a minimum value at $x = a, y = b$.

If $f(a, b) < f(a + h, b + k)$ for small values of h and k positive or negative.

Extreme Value : A maximum or a minimum value of a function is called an extreme value.

Art-5. Necessary Condition for Maxima and Minima

By applying Talylor's Theorem to the function $f(x, y)$ we have

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{\lfloor 2}\left(h\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)^2 + \ldots$$

$$\therefore \quad f(x + h, y + k) - f(x, y) = \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{\lfloor 2}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)^2 + \ldots$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

For small values of h and k, the first degree terms determine the sign of R.H.S. and consequently of L.H.S. of (1).

Since h, k are both positive or both negative.

$\therefore$ the change of the sign of h and k will affect the sign of L.H.S. of (1).

But for maximum and minimum $f(x+h, y+k) - f(x, y)$ must be of invariable sign for all arbitrary small values of h and k.

$\therefore$ necessary condition for maximum and minimum is

$$h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} = 0$$

ie At an extreame point (x,y)

Since this is true for all values of h, k

$$\therefore \quad \frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

are the required necessary conditions.

Note : $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ are necessary but not sufficient conditions.

Art-6. Sufficient Conditions for Maxima and Minima

Let $f(x, y)$ be a function of two variables x and y.

Let $\frac{\partial^2 f}{\partial x^2} = A$, $\frac{\partial^2 f}{\partial x \partial y} = B$, $\frac{\partial^2 f}{\partial y^2} = C$ when $x = a$, $y = b$.

Assume that $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

i.e. necessary conditions exist for maxima and minima.

$$\therefore \quad f(a+h, b+k) - f(a, b) = \frac{1}{\lfloor 2}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + R_3$$

$$\Rightarrow \quad f(a+h, b+k) - f(a, b) = \frac{1}{\lfloor 2}(Ah^2 + 2Bhk + Ck^2) + R_3 \qquad ...(1)$$

Since h and k are sufficiently small.

$\therefore$ the sign of R.H.S. and consequently of L.H.S. of (1) is determined by

$$Ah^2 + 2Bhk + Ck^2 = \frac{1}{A}(A^2h^2 + 2ABhk + ACk^2)$$

$$= \frac{1}{A}\left[(A^2h^2 + 2ABhk + B^2k^2) + (ACk^2 - B^2k^2)\right]$$

$$= \frac{1}{A}\left[(Ah + Bk)^2 + (AC - B^2)k^2\right] \qquad ...(2)$$

(*i*) When $AC - B^2 > 0$

$\therefore$ sign of R.H.S. of (2) is same as that of A.

(*ii*) When $AC - B^2 < 0$

$\therefore$ sign of R.H.S. of (2) is not certain

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Case I. When $AC - B^2 > 0$

Sub-case I : When $A > 0$

$\therefore$ L.H.S. of (1) > 0 *i.e.* $f(a+h,\ b+k) - f(a, b) > 0$

$\therefore$ $f(a+h, b+k) > f(a, b)$

$\Rightarrow$ f is minimum.

Sub-case II : When $A < 0$

$\therefore$ L.H.S. of (1) < 0

i.e. $f(a+h, b+k) - f(a, b) < 0$

$\therefore$ $f(a+h,\ b+k) < f(a, b)$

$\Rightarrow$ f is maximum.

Case II : When $AC - B^2 < 0$

$\therefore$ f is neither maximum nor minimum.

Case III : When $AC - B^2 = 0$ *i.e.* $AC = B^2$

$$\therefore \quad A h^2 + 2 B h k + C k^2 = \frac{1}{A}\left[A^2 h^2 + 2ABhk + ACk^2\right]$$

$$= \frac{1}{A}\left[A^2 h^2 + 2ABhk + B^2 k^2\right] \qquad [\because AC = B^2]$$

$$= \frac{1}{A}(Ah + Bk)^2$$

$\therefore$ sign of $A h^2 + 2 B h k + C k^2$ is same as that of A.

Sub-case I. When $A > 0$

$\therefore$ f is minimum.

Sub-case II. When $A < 0$

$\therefore$ f is maximum.

Case IV. When $AC - B^2 = 0$ and $A h + B k = 0$

$\therefore$ second degree terms of (1) vanish and so sign of R.H.S. of (1) depends upon third degree terms in h, k which change sign with the change of sign of h and k and so the sign of L.H.S. of (1) is will also change.

$\therefore$ f is neither maximum nor minimum.

Note. These sufficient conditions are also known as Lagrange's conditions.

Art-7. Working Method for Maxima and Minima

Let $f(x, y)$ be given function.

Step I. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$

Step II. Solve the equations $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$ simultaneously for x and y.

Let $(x_1, y_1), (x_2, y_2), \ldots$ be the points.

Step III. Find $A = \dfrac{\partial^2 f}{\partial x^2}, B = \dfrac{\partial^2 f}{\partial x \partial y}, C = \dfrac{\partial^2 f}{\partial y^2}$ and calculate values of A, B , C for each point.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Step IV. (*i*) If for a point (x_1, y_1), we have $AC - B^2 > 0$ and $A < 0$, then $f(x, y)$ is a **maxima** for this pair and maximum value is $f(x_1, y_1)$.

(*ii*) If for a point (x_1, y_1), we have

$AC - B^2 > 0$ and $A > 0$, then $f(x, y)$ is a **minima** for this pair and minimum value is $f(x_1, y_1)$

(*iii*) If $AC - B^2 < 0$ for a point (x_1, y_1), then there is neither max. nor minimum of $f(x, y)$. In this case $f(x, y)$ is said to have a **saddle point** at (x_1, y_1).

(*iv*) If $AC - B^2 = 0$ for some point (a, b), the case is doubtful

In this case,

if $f(a, b) - f(a + h, b + k) > 0$ for small values of h and k, positive or negative, then f is max. at (a, b),

if $f(a, b) - f(a + h, b + k) < 0$ for small values of h and k, positive or negative, then f is min. at (a, b),

if $f(a, b) - f(a + h, b + k)$ does not keep the same sign for small values of h and k, then there is neither max. nor minimum value.

Note. The points (x_1, y_1), (x_2, y_2), ... are called stationary or critical points and values of $f(x, y)$ at these points are called stationary values.

ILLUSTRATIVE EXAMPLES

Example 1. Find all the maxima and minima of the function

$$f(x, y) = x^3 + y^3 - 63(x + y) + 12xy.$$

Sol. Here $f(x, y) = x^3 + y^3 - 63(x + y) + 12xy$

Step I. $\dfrac{\partial f}{\partial x} = 3x^2 - 63 + 12y, \dfrac{\partial f}{\partial y} = 3y^2 - 63 + 12x$

Step II. Let us solve $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$

i.e., $3x^2 - 63 + 12y = 0$ and $3y^2 - 63 + 12x = 0$

or $x^2 + 4y - 21 = 0$ ...(1)

and $y^2 + 4x - 21 = 0$ ...(2)

Subtracting (2) from (1), we get,

$$x^2 - y^2 + 4y - 4x = 0$$

or $(x - y)(x + y) - 4(x - y) = 0$ or $(x - y)(x + y - 4) = 0$

Either $x - y = 0$

$\therefore$ $x = y$ ...(3)

Also $x^2 + 4y - 21 = 0$ ...(4)

From (3) and (4), we get

$x^2 + 4x - 21 = 0$

$\therefore$ $(x + 7)(x - 3) = 0$

$\therefore$ $x = 3, -7$

$\therefore$ from (3), $y = 3, -7$

or $x + y - 4 = 0$

$\therefore$ $x = 4 - y$ ...(5)

Also $x^2 + 4y - 21 = 0$ ... (6)

From (5) and (6), we get

$(4 - y)^2 + 4y - 21 = 0$

or $y^2 - 8y + 16 + 4y - 21 = 0$

or $y^2 - 4y - 5 = 0$

$\therefore$ $(y - 5)(y + 1) = 0$

$\therefore$ $y = 5, -1$

From (5), $x = -1, 5$

$\therefore$ four critical points are $(3, 3)$, $(-7, -7)$, $(-1, 5)$ and $(5, -1)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Step III. $A = \dfrac{\partial^2 f}{\partial x^2} = 6x$

$B = \dfrac{\partial^2 f}{\partial x \partial y} = 12$

$C = \dfrac{\partial^2 f}{\partial y^2} = 6y$

$\therefore \quad AC - B^2 = 36xy - 144$

Step IV. At (3, 3)

$AC - B^2 = 36(3)(3) - 144 = 324 - 144 = 180 > 0$

$A = 6(3) = 18 > 0$

$\therefore \quad f(x, y)$ is minimum at (3, 3)

and minimum value $= f(3, 3) = 27 + 27 - 63(3 + 3) + 12(3)(3)$

$= 27 + 27 - 378 + 108 = -216$

At (– 7, – 7)

$AC - B^2 = 36(-7)(-7) - 144 = 1764 - 144 = 1620 > 0$

Also $\quad A = 6(-7) = -42 < 0$

$\therefore \quad f(x, y)$ is maximum at (– 7, – 7)

and maximum value $= f(-7, -7) = (-7)^3 + (-7)^3 - 63(-7 - 7) + 12(-7)(-7)$

$= -343 - 343 + 882 + 588 = 784$

At (– 1, 5)

$AC - B^2 = 36(-1)(5) - 144 = -180 - 144 = -324 < 0$

$\therefore$ at (– 1, 5), $f(x, y)$ has neither max. nor minimum.

$\therefore$ (–1, 5) is a **saddle point**.

At (5, – 1)

$AC - B^2 = 36(5)(-1) - 144 = -180 - 144 = -324 < 0$

$\therefore$ at (5, – 1), $f(x, y)$ has neither maximum nor minimum.

$\therefore$ (5, – 1) is a **saddle point**.

Example 2. Show that (0, 0) is the only critical point of the function $f(x, y) = y^2 + x^2 y + x^4$ and that f has a minima at that point.

Sol. Given function is

$f(x, y) = y^2 + x^2 y + x^4$

Step I. $\dfrac{\partial f}{\partial x} = 2xy + 4x^3, \ \dfrac{\partial f}{\partial y} = 2y + x^2$

$\therefore$ (0, 0) is the only critical point of the given function.

Step II. Let us solve $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$

i.e. $2xy + 4x^3 = 0$ *i.e.* $x(y + 2x^2) = 0$ ...(1)

and $2y + x^2 = 0$ ...(2)

Solving (1) and (2), we get,

$x = 0, \ y = 0$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bhatinda, Jammu

Step III. $A = \dfrac{\partial^2 f}{\partial x^2} = 2y + 8x^2$

$$B = \frac{\partial^2 f}{\partial x \partial y} = 2x$$

$$C = \frac{\partial^2 f}{\partial y^2} = 2$$

$\therefore \quad AC - B^2 = (2y + 8x^2)(2) - (2x)^2$

Step IV. At (0, 0).

$AC - B^2 = 0$

$\therefore$ the standard criterion for maxima and minima fails

$\therefore$ it needs further investigation.

Consider $f(x, y) - f(0, 0) = (y^2 + x^2 y + x^4) - 0$

$$= \left(y + \frac{1}{2}x^2\right)^2 + \frac{3}{4}x^2 \geq 0 \ \forall (x, y) \in \mathbf{R}^2$$

$\therefore \quad f(x, y) \geq f(0, 0) \ \forall (x, y) \in \mathbf{R}^2$

i.e., (0, 0) is a point of minima for the given function f and the minimum value is zero.

Example 3. Find the local maxima, local minima and saddle point, if any , of the function

$$f(x, y) = 2xy - 5x^2 - 2y^2 + 4x - 4$$

Sol. Given function is

$$f(x, y) = 2xy - 5x^2 - 2y^2 + 4x - 4$$

Step I. $\dfrac{\partial f}{\partial x} = 2y - 10x + 4, \quad \dfrac{\partial f}{\partial y} = 2x - 4y$

Step II. Let us solve $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$

i.e. $\quad 2y - 10x + 4 = 0 \quad$ *i.e.* $5x - y - 2 = 0 \qquad$ ...(1)

and $\quad 2x - 4y = 0 \quad$ *i.e.* $x - 2y = 0 \qquad$ ...(2)

From (2), $\quad x = 2y$

$\therefore$ from (1), $10y - y - 2 = 0 \quad$ *i.e.* $y = \dfrac{2}{9}$

$\therefore \quad x = \dfrac{4}{9}$

$\therefore$ critical point is $\left(\dfrac{4}{9}, \dfrac{2}{9}\right)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Step III. $$A = \frac{\partial^2 f}{\partial x^2} = -10$$

$$B = \frac{\partial^2 f}{\partial x\, \partial y} = 2$$

$$C = \frac{\partial^2 f}{\partial y^2} = -4$$

$\therefore$ $$AC - B^2 = (-10)(-4) - (2)^2 = 40 - 4 = 36$$

At $\left(\frac{4}{9}, \frac{2}{9}\right)$,

$$AC - B^2 = 36 > 0$$
$$A = -10 < 0$$

$\therefore$ $f(x, y)$ is local maximum at $\left(\frac{4}{9}, \frac{2}{9}\right)$

Local maximum value $= 2\left(\frac{4}{9}\right)\left(\frac{2}{9}\right) - 5\left(\frac{4}{9}\right)^2 - 2\left(\frac{2}{9}\right)^2 + 4\left(\frac{4}{9}\right) - 4$

$$= \frac{16}{81} - \frac{80}{81} - \frac{8}{81} + \frac{16}{9} - 4 = \frac{16 - 80 - 8 + 144 - 324}{81}$$

$$= -\frac{252}{81} = -\frac{28}{9}$$

Example 4. Find the extreme value (if any) of the following functions :

(*i*) $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

(*ii*) $f(x, y) = 2x^4 - 3x^2 y + y^2$

Sol. (*i*) Here $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

Step I. $\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y$, $\frac{\partial f}{\partial y} = 4y^3 + 4x - 4y$

Step II. Let us solve $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

i.e., $x^3 - x + y = 0$ ...(1)

and $y^3 + x - y = 0$ ...(2)

Adding (1) and (2), $x^3 + y^3 = 0$ or $y^3 = -x^3$

$\therefore$ $y = -x$

$\therefore$ from (1), $x^3 - x - x = 0$ or $x^3 - 2x = 0$

$\Rightarrow$ $x(x^2 - 2) = 0$ $\Rightarrow$ $x = 0, \sqrt{2}, -\sqrt{2}$

$\therefore$ $y = 0, -\sqrt{2}, \sqrt{2}$

$\therefore$ critical points are $(0, 0)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bhatinda Punjab

Step III. $A = \dfrac{\partial^2 f}{\partial x^2} = 12x^2 - 4$

$$B = \frac{\partial^2 f}{\partial x \partial y} = 4$$

$$C = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4 \cdot$$

$\therefore \quad AC - B^2 = (12x^2 - 4)(12y^2 - 4) - 16$

Step IV. At (0, 0)

$AC - B^2 = (0 - 4)(0 - 4) - 16 = 0$

$\therefore$ at (0, 0), further investigation is needed.

Consider $f(a, b) - f(a + h, b + k) = f(0, 0) - f(h, k)$

$$= 0 - (h^4 + k^4 - 2h^2 + 4hk - 2k^2)$$
$$= -h^4 - k^4 + 2h^2 + 2k^2 - 4hk$$
$$= 2(h - k)^2 - (h^4 + k^4)$$

which does not keep the same sign for all small values of h and k

$\therefore$ there is neither maximum nor minimum at (0, 0).

At $(\sqrt{2}, -\sqrt{2})$

$AC - B^2 = (12 \times 2 - 4)(12 \times 2 - 4) - 16 = 400 - 16 = 384 > 0$

Also $A = 12 \times 2 - 4 = 20 > 0$

$\therefore$ $f(x, y)$ has a minimum value at $(\sqrt{2}, -\sqrt{2})$

and minimum value $= (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4(-\sqrt{2})(\sqrt{2}) - 2(\sqrt{2})^2$

$$= 4 + 4 - 4 - 8 - 4 = -8$$

At $(-\sqrt{2}, \sqrt{2})$

$AC - B^2 = (12 \times 2 - 4)\ (12 - 2) - 16 = 20 > 0$

Also $A = 12 \times 2 - 4 = 20 > 0$

$\therefore$ $f(x, y)$ has a minimum value at $(-\sqrt{2}, \sqrt{2})$

and minimum value $= (-\sqrt{2})^4 + (\sqrt{2})^4 - 2(-\sqrt{2})^2 + 4(-\sqrt{2})(\sqrt{2}) - 2(\sqrt{2})^2$

$$= 4 + 4 - 4 - 8 - 4 = -8$$

(*ii*) Here $f(x, y) = 2x^4 - 3x^2y + y^2$

Step I. $\dfrac{\partial f}{\partial x} = 8x^3 - 6xy, \ \dfrac{\partial f}{\partial y} = -3x^2 + 2y$

Step II. Let us solve $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$

i.e., $8x^3 - 6xy = 0$ ...(1)

and $3x^2 - 2y = 0$ ...(2)

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

 Gandhi Memorial College Of Education Bantalab Jammu

From (1), $2x(4x^2 - 3y) = 0$

$\Rightarrow \quad x = 0, x^2 = \dfrac{3y}{4}$

When $x = 0$, from (2),

$0 - 2y = 0$

$\therefore \quad y = 0$

$\therefore$ point is (0, 0)

When $x^2 = \dfrac{3y}{4}$, from (2),

$3 \times \dfrac{3y}{4} - 2y = 0$

$\Rightarrow \quad y = 0$

$\therefore \quad x = 0$

$\therefore$ point is (0, 0)

$\therefore$ (0, 0) is the only critical point.

Step III. $A = \dfrac{\partial^2 f}{\partial x^2} = 24x^2 - 6y$

$B = \dfrac{\partial^2 f}{\partial x \partial y} = -6x$

$C = \dfrac{\partial^2 f}{\partial y^2} = 2$

$\therefore \quad AC - B^2 = (24x^2 - 6y)(2) - (-6x)^2.$

Step IV. At (0, 0)

$AC - B^2 = (0 - 0)(2) - (-0)^2 = 0$

$\therefore$ at (0, 0) further investigation is needed.

Now consider $f(a, b) - f(a + h, b + k) = f(0, 0) - f(h, k)$

$$= 0 - (2h^4 - 3h^2 k + k^2) = -2h^4 + 3h^2 k - k^2$$

$$= \frac{1}{8}k^2 - 2\left(h^2 - \frac{3k}{4}\right)^2$$

which does not keep the same sign for all small values of h and k

$\therefore$ there is neither maximum nor minimum at (0, 0).

Example 5. Find the extreme values (if any) of the following function :

$$f(x, y) = x^3 y^2 (1 - x - y)$$

Sol. Here $f(x, y) = x^3 y^2 (1 - x - y) = x^3 y^2 - x^4 y^2 - x^3 y^3.$

Step I. $\dfrac{\partial f}{\partial x} = 3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3,$

$\dfrac{\partial f}{\partial y} = 2x^3 y - 2x^4 y - 3x^3 y^2$

Step II. Let us solve $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$

i.e., $3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3 = 0$ ...(1)

and $2x^3 y - 2x^4 y - 3x^3 y^2 = 0$ ...(2)

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College of Education Bantalab Jammu

From (1), $x^2\ y^2\ (3 - 4\,x - 3\,y) = 0$

From (2), $x^3\,y\,(2 - 2\,x - 3\,y) = 0$

Solving these, we get, two points as (0, 0), $\left(\frac{1}{2}, \frac{1}{3}\right)$ which are critical points.

Step III. $A = \dfrac{\partial^2 f}{\partial x^2} = 6\,x\,y^2 - 12\,x^2\,y^2 - 6\,x\,y^3$

$$B = \frac{\partial^2 f}{\partial x\,\partial y} = 6\,x^2\,y - 8\,x^3\,y - 9\,x^2\,y^2$$

$$C = \frac{\partial^2 f}{\partial y^2} = 2\,x^3 - 2\,x^4 - 6\,x^3\,y$$

Step IV. At (0, 0)

$AC - B^2 = (0)\,(0) - (0) = 0$

$\therefore$ further investigation is needed at (0, 0).

Now $f(0, 0) - f(h, k) = -\,h^3\,k^2\,(1 - h - k) = -\,h^3\,k^2 + h^4\,k^2 + h^3\,k^3$

$= -\,h^3\,k^2$ (neglecting $h^4\,k^2 + h^3\,k^3$ which is small as compared to $h^3\,k^2$)

$$= \begin{cases} < 0 & \text{, if } h > 0 \\ > 0 & \text{, if } h < 0 \end{cases}$$

$\therefore$ $f(0, 0) - f(h, k)$ does not keep the same sign for all small values of h and k.

$\therefore$ there is no extreme value at (0, 0).

At $\left(\frac{1}{2}, \frac{1}{3}\right)$

$$AC - B^2 = \left[6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^2 - 12\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2 - 6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^3\right]$$

$$\times\left[2\left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{2}\right)^4 - 6\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)\right]$$

$$-\left[6\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right) - 8\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right) - 9\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2\right]^2 = \frac{1}{72} - \frac{1}{144} = \frac{1}{144} > 0$$

Also $A = 6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^2 - 12\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2 - 6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^3 = -\,\frac{1}{9} < 0$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$\therefore \quad f(x, y)$ has a max. value at $\left(\frac{1}{2}, \frac{1}{3}\right)$

and max. value $= \frac{1}{8} \times \frac{1}{9} \times \left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{432}$

Example 6. Show that of all triangles inscribed in a circle, the one with maximum area is equilateral.

Sol. Let r be the radius of a circle and ABC be a triangle inscribed in it with angles x, y and z.

$\therefore \quad 0 < x, y, z < \pi$ and $x + y + z = \pi$ ...(1)

If S is the area of ΔABC, then

$S = \frac{1}{2} a\, b \sin z$ where a, b are the lengths of the sides of Δ ABC containing the angle z.

Now $a = 2r \sin x$, $b = 2r \sin y$

$\therefore \quad S = \frac{1}{2}(2r \sin x)\ (2r \sin y) \sin z$

$= 2r^2 \sin x \sin y \sin (x + y)$ [$\because$ of (1)]

$\Rightarrow \quad \frac{S}{2r^2} = f$ (say) $= \sin x \sin y \sin (x + y)$

(We are to maximise it)

Let $f(x, y) = \sin x \sin y \sin (x + y)$

$f_x = \sin y\, [\sin x . \cos (x + y) + \sin (x + y) \cos x]$

$= \sin y \sin (2x + y)$

$f_y = \sin x\, [\sin y \cos (x + y) + \sin (x + y) \cos y]$

$= \sin x \sin (x + 2y)$

$f_{xx} = 2 \sin y \cos (2x + y)$, $f_{yy} = 2 \sin x \cos (x + 2y)$

$f_{xy} = \sin y \cos (2x + y) + \sin (2x + y) \cos y$

$= \sin (y + 2x + y) = \sin (2x + 2y)$

For extreme values, $f_x = 0$ and $f_y = 0$

i.e., $\sin y \sin (2x + y) = 0$ and $\sin x \sin (x + 2y) = 0$

But $\sin x, \sin y \neq 0$ as $0 < x, y < \pi$

$\therefore \quad \sin (2x + y) = 0$ and $\sin (x + 2y) = 0$

$\Rightarrow \quad 2x + y = \pi$ and $x + 2y = \pi$

$\Rightarrow \quad x = y = \frac{\pi}{3}, \quad \therefore \ z = \frac{\pi}{3}.$

At the point $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $A = -\sqrt{3}, B = -\frac{\sqrt{3}}{2}, C = -\sqrt{3}$

$\therefore \quad AC - B^2 = \left(-\sqrt{3}\right)\left(-\sqrt{3}\right) - \left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{9}{4} > 0$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Also $A < 0$

$\therefore \quad \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is a point of maxima

Hence area of ΔABC is maximum when

$$x = \frac{\pi}{3}, y = \frac{\pi}{3} \text{ and } z = \frac{\pi}{3}$$

i.e. the ΔABC is an equilateral triangle.

Example 7. A rectangular box, open at the top, is to have volume of 32 cubic metres. Find the dimensions of the box requiring least material for its construction.

Sol. Let x, y, z metres be the edges of the open box and S be its surface

$\therefore \quad S = xy + 2yz + 2zx$ ...(1)

Also volume = 32 cubic metres

$\therefore \quad xyz = 32$

$\therefore \quad z = \frac{32}{xy}$ ...(2)

$\therefore$ from (1), $S = xy + 2y \cdot \frac{32}{xy} + 2x \cdot \frac{32}{xy}$

$\therefore \quad S = xy + \frac{64}{x} + \frac{64}{y}$

Step I. $\frac{\partial S}{\partial x} = y - \frac{64}{x^2}, \frac{\partial S}{\partial y} = x - \frac{64}{y^2}$

Step II. Let us solve $\frac{\partial S}{\partial x} = 0$ and $\frac{\partial S}{\partial y} = 0$

i.e., $y - \frac{64}{x^2} = 0$ ...(3)

and $x - \frac{64}{y^2} = 0$ ...(4)

From (3) $y = \frac{64}{x^2}$

$\therefore$ from (4), $x - \frac{64}{(64)^2} \times x^4 = 0$

or $(64x - x^4) = 0$

$\therefore \quad x(64 - x^3) = 0$

$\therefore \quad x = 0, 4$

Now $x = 0$ does not give any finite value of y

When $x = 4$, $y = \frac{64}{16} = 4$

$\therefore$ critical point is (4, 4)

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Step III. $A = \dfrac{\partial^2 S}{\partial x^2} = \dfrac{128}{x^3}$

$$B = \frac{\partial^2 S}{\partial x\, \partial y} = 1$$

$$C = \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3}$$

$$\therefore \quad AC - B^2 = \frac{(128)^2}{x^3\, y^3} - 1$$

Step IV. At (4, 4)

$$AC - B^2 = \frac{(128)^2}{64 \times 64} - 1 = 4 - 1 = 3 > 0$$

Also $A = \dfrac{128}{4^3} = 2 > 0$

$\therefore$ S is least when $x = 4, y = 4$

and $\therefore$ from (2), $z = \dfrac{32}{4 \times 4} = 2$

$\therefore$ dimensions of the box requiring least material are 4 metres, 4 metres and 2 metres.

EXERCISE 14 (*a*)

1. Find the extreme values of the function
$$f(x, y) = x^3 + y^3 + 3\,x\,y$$
2. Find the extreme values of the function
$$f(x, y) = y^2 - x\,y + x^2 - 2\,y + 3\,x + 7$$
3. Find the extreme values of the function
$$f(x, y) = 3\,x^2 + 4\,x\,y + y^2 + x^3.$$
4. Find all the critical points of the function $f(x, y) = x^2 + x\,y$ and examine for maxima, minima or neither.
5. Find the extreme values of the following functions :
   (*i*) $x^3 + 8\,y^3 + 6\,x\,y$
   (*ii*) $12\,x^2 + 8\,x\,y + y^2 + 8\,x^3$
6. Find points of extreme values, if any, of the function f defined by
$$f(x, y) = x^3 + 3\,x + y^3 - y + 4.$$
7. Find the extreme value (if any) of the following functions :
   (*i*) $f(x, y) = x^2 + x\,y + 3\,x + 2\,y + 5$
   (*ii*) $f(x, y) = x^3 + y^3 - 3\,a\,x\,y$
   (*iii*) $f(x, y) = x^2 + x\,y + y^2 + x + y$
   (*iv*) $f(x, y) = x^3 + y^3 - 3\,x - 12\,y + 20$
   (*v*) $f(x, y) = 2\,x^3 + 3\,x\,y^2 + 5\,x^2 + y^2$
   (*vi*) $f(x, y) = x^3 - 3\,x^2 - 4\,y^2 + 1$
   (*vii*) $f(x, y) = x^3 + 3\,x\,y^2 - 15\,x^2 - 15\,y^2 + 72\,x.$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

8. (*a*) Find the extreme values (if any) of the following functions :

(*i*) $f(x, y) = (x - y)^4 + (y - 1)^4$

(*ii*) $f(x, y) = x\,y\,(a - x - y)$

(*b*) Show that $f(x, y) = (y - x)^4 + (x + 2)^4$ has a minima at (2, 2)

9. Discuss $f(x, y) = y^2 + x^2y + a\,x^4$ for extreme considering all possible values of a.

10. Find the maximum and minimum values of the following functions

(*i*) $\sin x \sin y \sin (x + y)$

(*ii*) $\sin x + \sin y + \sin (x + y)$

(*iii*) $\sin x + \sin y + \cos (x + y)$

ANSWERS

1. Neither max. nor min. at (0, 0) ; max. value 1 at (–1, –1).

2. Minimum value $\frac{14}{3}$ at $\left(-\frac{4}{3}, \frac{1}{3}\right)$

3. Neither max. nor min. at (0, 0) ; min. value $-\frac{4}{27}$ at $\left(\frac{2}{3}, -\frac{4}{3}\right)$.

4. Neither max. nor min. at (0, 0).

5. (*i*) Neither max. nor min. at (0, 0)

(*ii*) Neither max. nor min. at (0, 0) ; min. value $-\frac{4}{27}$ at $\left(\frac{1}{3}, -\frac{4}{3}\right)$

6. No point of extreme value.

7. (*i*) Neither max. nor min. at (–2, 1)

(*ii*) Neither max. nor min. at (0, 0) ; Min. value = $-a^3$ at (a, a) for $a > 0$ and Max. value = $-a^3$ at (a, a) for $a < 0$.

(*iii*) Minimum value = $-\frac{1}{3}$ at $\left(-\frac{1}{3}, -\frac{1}{3}\right)$

(*iv*) Min. value = 2 at (1, 2) ; Neither max. nor min. at (1, – 2) ; Max. value = 38 at (–1, –2).

(*v*) Minimum value = 0 at (0, 0) ; maximum value = $\frac{125}{27}$ at $\left(-\frac{5}{3}, 0\right)$

(*vi*) Maximum value = 1 at (0, 0)

(*vii*) Max. value = 112 at (4, 0) ; min. value = 108 at (6, 0) ; Neither max. nor min. at (5, 1) and (5, –1).

8. (*a*) (*i*) Min. value 0 at (1, 1)

(*ii*) No extreme value at (0, 0) , $(a, 0)$, $(0, a)$;

max. value = $\frac{a^3}{27}$ when $a > 0$, min. value = $\frac{a^3}{27}$ when $a < 0$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

9. If $a \geq \frac{1}{4}$, then (0, 0) is a point of minima ;

if $a < \frac{1}{4}$, then (0, 0) is neither *a* point of maxima nor minima

10. (*i*) Max. value $= \frac{3\sqrt{3}}{8}$ at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ (*ii*) Max. value $= \frac{3\sqrt{3}}{8}$ at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

(*iii*) Neither max. nor min. at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$; Minima at $\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$ and maxima at

$$x = y = n\pi + (-1)^n \frac{\pi}{6}, n \in \mathbf{I}$$

Art-8. Working Method for Maxima and Minima of a Function $f(x, y, z)$

Step I. Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$

Step II. Solve $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial z} = 0$ simultaneously to get critical points.

Step III $A = \frac{\partial^2 f}{\partial x^2}$, $B = \frac{\partial^2 f}{\partial y^2}$, $C = \frac{\partial^2 f}{\partial z^2}$,

$F = \frac{\partial^2 f}{\partial y \partial z}$, $G = \frac{\partial^2 f}{\partial z \partial x}$, $H = \frac{\partial^2 f}{\partial x \partial y}$,

and evaluate the three expressions

$$A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} \text{ in this order}$$

Step IV. Find the value of each of the three expressions at a stationary point.

If the three expressions are alternatively negative and positive, then $f(x, y, z)$ is maximum.

If the three expressions are all positive, then $f(x, y, z)$ is minimum.

If we fail to get any idea of maximum value or minimum value from the expressions

$$A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}, \text{ then we calculate } d^2F = \Sigma F_{xx} (dx)^2 + 2 \Sigma F_{yz}\, dy\, dz.$$

If d^2F is negative definite at (a, b, c), then (a, b, c) is a point of maxima and if d^2F is positive definite at (a, b, c), then (a, b, c) is a point of minima. If however d^2F is indefinite *i.e.*, it can have positive as well as negative values, then F has neither maxima nor minima at (a, b, c).

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bhatinda, Amritsar

ILLUSTRATIVE EXAMPLES

Example 1. Show that $u = (x+y+z)^3 - 3(x+y+z) - 24xyz + a^3$ has a minima at (1, 1, 1) and a maxima at (– 1, – 1, – 1).

Sol. Here $u = (x+y+z)^3 - 3(x+y+z) - 24xyz + a^3$

Step I. $\dfrac{\partial u}{\partial x} = 3(x+y+z)^2 - 3 - 24yz$

$$\frac{\partial u}{\partial y} = 3(x+y+z)^2 - 3 - 24zx$$

$$\frac{\partial u}{\partial z} = 3(x+y+z)^2 - 3 - 24xy$$

Step II. Let us solve $\dfrac{\partial u}{\partial x} = 0$, $\dfrac{\partial u}{\partial y} = 0$ and $\dfrac{\partial u}{\partial z} = 0$ for x, y, z.

i.e., $3(x+y+z)^2 - 3 - 24yz = 0$

$3(x+y+z)^2 - 3 - 24zx = 0$

$3(x+y+z)^2 - 3 - 24xy = 0$

or $(x+y+z)^2 - 1 = 8yz$ ...(1)

$(x+y+z)^2 - 1 = 8zx$ ...(2)

$(x+y+z)^2 - 1 = 8xy$ ...(3)

From (1), (2), (3), we get,

$8yz = 8zx = 8xy$

or $\dfrac{1}{x} = \dfrac{1}{y} = \dfrac{1}{z}$

or $x = y = z$

$\therefore$ from (1), $9x^2 - 1 = 8x^2$

or $x^2 = 1$, $\therefore$ $x = 1, -1$

$\therefore$ $y = 1, -1$; $z = 1, -1$

$\therefore$ two critical points are (1, 1, 1), (– 1, – 1, – 1).

Step III. $A = \dfrac{\partial^2 u}{\partial x^2} = 6(x+y+z)$

$$B = \frac{\partial^2 u}{\partial y^2} = 6(x+y+z)$$

$$C = \frac{\partial^2 u}{\partial z^2} = 6(x+y+z)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$F = \frac{\partial^2 u}{\partial y\, \partial z} = 6\,(x + y + z) - 24\,x$$

$$G = \frac{\partial^2 u}{\partial z\, \partial x} = 6\,(x + y + z) - 24\,y$$

$$H = \frac{\partial^2 u}{\partial x\, \partial y} = 6\,(x + y + z) - 24\,z$$

Step IV. At (1, 1, 1)

$A = 6\,(1 + 1 + 1) = 18,\ B = 18,\ C = 18$

$F = 6\,(1 + 1 + 1) - 24\,(1) = -6,\ G = -6,\ H = -6$

$$\begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 18 & -6 \\ -6 & 18 \end{vmatrix} = 324 - 36 = 288$$

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 18 & -6 & -6 \\ -6 & 18 & -6 \\ -6 & -6 & 18 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -6 \\ -24 & 24 & -6 \\ 48 & -24 & 18 \end{vmatrix}, \text{ by } C_1 + 3\,C_3,\ C_2 - C_3$$

$$= -6\begin{vmatrix} -24 & 24 \\ 48 & -24 \end{vmatrix} = -6 \times 24 \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = -144\,(1 - 2) = 144$$

$\therefore$ u is minimum at (1, 1, 1).

At (– 1, – 1, – 1)

$A = 6\,(-1 - 1 - 1) = -18,\ B = -18,\ C = -18$

$F = 6\,(-1 - 1 - 1) - 24\,(-1) = -18 + 24 = 6;\ G = 6,\ H = 6$

$$\begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} -18 & 6 \\ 6 & -18 \end{vmatrix} = 324 - 36 = 288$$

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} -18 & 6 & 6 \\ 6 & -18 & 6 \\ 6 & 6 & -18 \end{vmatrix}$$

$$= 216\begin{vmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -3 \end{vmatrix} = 216\begin{vmatrix} 0 & 0 & 1 \\ 4 & -4 & 1 \\ -8 & 4 & -3 \end{vmatrix}, \text{ by } C_1 + 3\,C_3,\ C_2 - C_3$$

$$= 216\begin{vmatrix} 4 & -4 \\ -8 & 4 \end{vmatrix} = 216\,(16 - 32) = 216\,(-16) = -3456$$

$\therefore$ u has a maximum value at (– 1, – 1, – 1).

Example 2. Examine for extreme values the function

$$f(x, y, z) = x^2 + y^2 + z^2 + 2\,x\,y\,z$$

Sol. Here $f(x, y, z) = x^2 + y^2 + z^2 + 2\,x\,y\,z$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Ranchi Jammu

Step I. $\dfrac{\partial f}{\partial x} = 2x + 2yz$

$\dfrac{\partial f}{\partial y} = 2y + 2zx$

$\dfrac{\partial f}{\partial z} = 2z + 2xy$

Step II. Let us solve $\dfrac{\partial f}{\partial x} = 0$, $\dfrac{\partial f}{\partial y} = 0$ and $\dfrac{\partial f}{\partial z} = 0$ for x, y, z.

i.e., $yz = -x$ ...(1)

$zx = -y$ ...(2)

$xy = -z$ ...(3)

Multiply (1), (2), (3), we get,

$x^2y^2z^2 = -xyz$ or $xyz(xyz + 1) = 0$

$\therefore$ either $x = y = z = 0$

or $xyz = -1$

$\therefore$ from (1), (2), (3), we get,

$-x^2 = -1, -y^2 = -1, -z^2 = 1$

or $x = \pm 1, y = \pm 1, z = \pm 1$

$\therefore$ possible sets of values satisfying

$xyz = -1$ are $(-1, 1, 1), (1, -1, 1), (1, 1, -1), (-1, -1, -1)$

$\therefore$ critical points are

$(0, 0, 0), (-1, 1, 1), (1, -1, 1), (1, 1, -1), (-1, -1, -1)$.

Step III. $A = \dfrac{\partial^2 f}{\partial x^2} = 2, B = \dfrac{\partial^2 f}{\partial y^2} = 2, C = \dfrac{\partial^2 f}{\partial z^2} = 2$

$F = \dfrac{\partial^2 f}{\partial y\,\partial z} = 2x,\ G = \dfrac{\partial^2 f}{\partial z\,\partial x} = 2y,\ H = \dfrac{\partial^2 f}{\partial x\,\partial y} = 2z$

Step IV. At (0, 0, 0)

$A = 2, B = 2, C = 2$

$F = 0, G = 0, H = 0$

$$\begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$$

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8$$

$\therefore$ f has a local minimum at (0, 0, 0).

At (– 1, 1, 1)

$A = 2, B = 2, C = 2$

$F = -2, G = 2, H = 2$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} = 4 - 4 = 0$$

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 2 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 2 & 2 \\ 0 & 0 & -4 \\ 0 & -4 & 0 \end{vmatrix}, \text{ by } R_2 - R_1, R_3 - R_1$$

$$= 2\,(0 - 16) = -32$$

$\therefore$ three expressions A, $\begin{vmatrix} A & H \\ H & B \end{vmatrix}$, $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$ are neither all positive nor alternatively negative and positive.

$\therefore$ $f(x, y, z)$ has neither maximum nor minimum values at $(-1, 1, 1)$.

At $(1, -1, 1)$

$A = B = C = 2$

$F = 2,\ G = -2,\ H = 2.$

$$\begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} = 0$$

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 2 & 2 & -2 \\ 2 & 2 & 2 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 2 & 0 & 4 \\ -2 & 4 & 0 \end{vmatrix} = 2\,(0 - 16) = -32$$

$\therefore$ three expressions A, $\begin{vmatrix} A & H \\ H & B \end{vmatrix}$, $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$ are neither all positive nor alternatively negative and positive.

$\therefore$ $f(x, y, z)$ has neither maximum nor minimum value at $(1, -1, 1)$.

At (1, 1, – 1)

$A = B = C = 2$

$F = 2, G = 2, H = -2$

$$\begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix} = 4 - 4 = 0$$

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 2 & -2 & 2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ -2 & 0 & 4 \\ -2 & 4 & 0 \end{vmatrix} = 2\,(0 - 16) = -32$$

$\therefore$ as explained above, $f(x, y, z)$ is neither maximum nor minimum at $(1, 1, -1)$.

At (– 1, – 1, – 1)

$A = 2, B = 2, C = 2;\quad F = -2, G = -2, H = -2$

$$\begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix} = 4 - 4 = 0$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ -2 & 0 & -4 \\ 2 & -4 & 0 \end{vmatrix} = 2\,(0-16) = -32$$

$\therefore$ as explained above, $f(x, y, z)$ is neither maximum nor minimum at $(-1, -1, -1)$.

EXERCISE 14 (*b*)

1. Examine for extreme values the function
$$f(x, y, z) = 2x^2 + 3y^2 + 4z^2 - 3xy + 8z.$$
2. Examine for extreme values the function
$$f(x, y, z) = 2xyz - 4zx - 2yz + x^2 + y^2 + z^2 - 2x - 4y + 4z$$
3. Show that when $x = y = z = 1$, the function
$$f(x, y, z) = x^4 + y^4 + z^4 - 4xyz$$
has a stationary value and investigate its nature.
4. Examine for extreme values the function
$$f(x, y, z) = x^2 + y^2 + z^2 + x - 2z - xy.$$

ANSWERS

1. Minimum value $= -4$ at $(0, 0, -1)$
2. Minimum at $(1, 2, 0)$;
neither max. nor min. at $(2, 3, -1)$, $(2, 1, 1)$, $(0, 1, -1)$, $(0, 3, 1)$.
3. $(1, 1, 1)$ is a point of minima
4. Minimum value $= -\dfrac{4}{3}$ at $\left(-\dfrac{2}{3}, -\dfrac{1}{3}, 1\right)$

Art-9. Lagrange's Method of Undetermined Multipliers

Let $f(x, y, z)$ be a function of x, y, z which is to be examined for maximum or minimum value and let the variable be connected by the relation

$$\phi(x, y, z) = 0 \qquad \ldots(1)$$

Since $f(x, y, z)$ is to have a maximum or minimum value

$$\therefore \quad \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \qquad \ldots(2)$$

Differentiating (1), we get,

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0 \qquad \ldots(3)$$

Multiplying (2) by 1, (3) by λ and adding, we get,

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

For this equation to be satisfied identically, coeffs. of dx, dy, dz should be separately zero.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\therefore \quad \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \qquad \ldots(4)$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \qquad \ldots(5)$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \qquad \ldots(6)$$

Equations (1), (4), (5) and (6) give us the values of x, y, z, λ for which $f(x, y, z)$ is maximum or minimum.

Note. We explain the method with the help of examples.

ILLUSTRATIVE EXAMPLES

Example 1. Find the maximum and minimum value of $x^2 + y^2$ subject to the condition $3x^2 + 4xy + 6y^2 = 140$.

Sol. Let $f(x, y) = x^2 + y^2$

The constraint is

$$3x^2 + 4xy + 6y^2 = 140 \qquad \ldots(1)$$

Let $F(x, y) = x^2 + y^2 + \lambda(3x^2 + 4xy + 6y^2 - 140)$ where λ is Lagrange's multiplier.

For extreme points,

$$\frac{\partial F}{\partial x} = 2x + \lambda(6x + 4y) = 0$$

$$\frac{\partial F}{\partial y} = 2y + \lambda(4x + 12y) = 0$$

or $\quad (1 + 3\lambda)x + 2\lambda y = 0 \qquad \ldots(2)$

and $\quad 2\lambda x + (1 + 6\lambda)y = 0 \qquad \ldots(3)$

Since x and y are both non-zero

$$\therefore \quad \begin{vmatrix} 1+3\lambda & 2\lambda \\ 2\lambda & 1+6\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1 + 3\lambda)(1 + 6\lambda) - 4\lambda^2 = 0$$

$$\Rightarrow \quad 14\lambda^2 + 9\lambda + 1 = 0$$

$$\Rightarrow \quad \lambda = \frac{-9 \pm \sqrt{81 - 56}}{28} = \frac{-9 \pm 5}{28} = -\frac{1}{2}, -\frac{1}{7}$$

Take $\lambda = -\frac{1}{2}$

$\therefore$ from (2), $x = -2y$

$\therefore$ from (1), $12y^2 - 8y^2 + 6y^2 - 140 = 0 \quad \Rightarrow y^2 = 14$

$\therefore \quad x^2 = 4y^2 = 56$

$\therefore \quad x^2 + y^2 = 56 + 14 = 70$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Take $\lambda = -\dfrac{1}{7}$

$\therefore$ from (2), $y = 2x$

$\therefore$ from (1), $3x^2 + 8x^2 + 24x^2 = 140 \Rightarrow x^2 = 4$

$\therefore$ $y^2 = 4x^2 = 16$

$\therefore$ $x^2 + y^2 = 16 + 4 = 20$

Hence maximum value of $x^2 + y^2$ is 70 and minimum value is 20.

Example 2. Let $f: \mathbf{R}^3 \to \mathbf{R}$ be defined as $f(x, y, z) = xyz$. Determine x, y, z for maximum of f where $xy + 2yz + 2zx = 108$.

Sol. Here $f(x, y, z) = xyz$ ...(1)

The constraint is

$$xy + 2yz + 2zx - 108 = 0 \quad ...(2)$$

Let $F(x, y, z) = xyz + \lambda(xy + 2yz + 2zx - 108)$

where λ is Lagrange's multiplier to be determined.

For extreme points,

$$F_x = yz + \lambda(y + 2z) = 0 \quad ...(3)$$

$$F_y = zx + \lambda(x + 2z) = 0 \quad ...(4)$$

$$F_z = xy + \lambda(2y + 2x) = 0 \quad ...(5)$$

Multiplying (3) by x and (4) by y, we get,

$$xyz + \lambda(xy + 2zx) = 0$$

and $xyz + \lambda(xy + 2yz) = 0$

Subtracting, $\lambda(2zx - 2yz) = 0$

$\therefore$ $z(x - y) = 0 \Rightarrow x = y$ as $z \neq 0$

Multiplying (4) by y and (5) by z, we get,

$$xyz + \lambda(xy + 2yz) = 0$$

$$xyz + \lambda(2yz + 2zx) = 0$$

Subtracting, $\lambda(xy - 2zx) = 0$

$\therefore$ $x(y - 2z) = 0 \Rightarrow y = 2z$ as $x \neq 0$

Putting $x = y = 2z$ in (2), we get

$$2z \cdot 2z + 2 \cdot 2z \cdot z + 2z \cdot 2z - 108 = 0$$

$\therefore$ $12z^2 = 108 \Rightarrow z^2 = 9 \Rightarrow z = 3$ as z is positive

$\therefore$ $x = 6,\ y = 6,\ z = 3$

$\therefore$ for maximum value of xyz, we have,

$$x = 6,\ y = 6,\ z = 3$$

Example 3. Find the extreme values of the function $x^2 + y^2 + z^2$ subject to the condition $lx + my + nz = p$.

Sol. We want to find the extreme value of

$$u = x^2 + y^2 + z^2 \quad ...(1)$$

when $lx + my + nz = p$ ...(2)

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Consider the function

$$F(x, y, z) = x^2 + y^2 + z^2 + \lambda (l x + m y + n z - p)$$

where λ is Lagrange's multiplier to be determined.

For extreme values,

$$F_x = 2x + \lambda l = 0$$

$$F_y = 2y + \lambda m = 0$$

$$F_z = 2z + \lambda n = 0$$

$$\Rightarrow \quad x = -\frac{\lambda l}{2}, y = -\frac{\lambda m}{2}, z = -\frac{\lambda n}{2}$$

$\therefore$ from (2), we get,

$$l\left(-\frac{\lambda l}{2}\right) + m\left(-\frac{\lambda m}{2}\right) + n\left(-\frac{\lambda n}{2}\right) = p$$

$$\therefore \quad -\lambda (l^2 + m^2 + n^2) = 2p$$

$$\Rightarrow \quad \lambda = -\frac{2p}{l^2 + m^2 + n^2}$$

$$\therefore \quad x = \frac{lp}{l^2 + m^2 + n^2}, y = \frac{mp}{l^2 + m^2 + n^2}, z = \frac{np}{l^2 + m^2 + n^2}$$

$\therefore$ stationary point is $\left(\frac{lp}{l^2 + m^2 + n^2}, \frac{mp}{l^2 + m^2 + n^2}, \frac{np}{l^2 + m^2 + n^2}\right)$

At this point,

$$d^2 F = \Sigma F_{xx} (dx)^2 + 2 \Sigma F_{yz}\, dy\, dz$$

$$= 2 [(dx)^2 + (dy)^2 + (dz)^2] \qquad [\because F_{xx} = F_{yy} = F_{zz} = 2, F_{xy} = F_{yz} = F_{zx} = 0]$$

which is positive definite

$\therefore$ u has a minimum value

$$= \frac{l^2 p^2}{(l^2 + m^2 + n^2)^2} + \frac{m^2 p^2}{(l^2 + m^2 + n^2)^2} + \frac{n^2 p^2}{(l^2 + m^2 + n^2)^2}$$

$$= \frac{p^2 (l^2 + m^2 + n^2)}{(l^2 + m^2 + n^2)^2} = \frac{p^2}{l^2 + m^2 + n^2}$$

Example 4. Prove that the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ is } \frac{8abc}{3\sqrt{3}}.$$

Sol. Let V be the volume of the rectangular parallelopiped whose edges are $2x, 2y, 2z$ where $x, y, z > 0$.

We have to find the maximum value of

$$V = 8xyz \qquad \ldots(1)$$

$$\text{when } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \qquad \ldots(2)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Consider the function

$$F(x, y, z) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

where λ is Lagrange's multiplier to be determined.

For extreme points,

$$F_x = 8yz + \lambda.\frac{2x}{a^2} = 0 \qquad \ldots(3)$$

$$F_y = 8zx + \lambda.\frac{2y}{b^2} = 0 \qquad \ldots(4)$$

$$F_z = 8xy + \lambda.\frac{2z}{c^2} = 0 \qquad \ldots(5)$$

Multiplying (3) by x, (4), by y, (5) by z and adding, we get

$$24xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0$$

or $\quad 24xyz + 2\lambda(1) = 0 \qquad [\because \text{ of } (2)]$

$\Rightarrow \quad \lambda = -12xyz$

$\therefore \quad$ from (3), $\quad 8yz - \dfrac{24x^2yz}{a^2} = 0$

$\Rightarrow x^2 = \dfrac{a^2}{3} \qquad \Rightarrow x = \dfrac{a}{\sqrt{3}} \qquad (\because x > 0)$

Similarly $y = \dfrac{b}{\sqrt{3}}, z = \dfrac{c}{\sqrt{3}}, \lambda = -\dfrac{4abc}{\sqrt{3}}$

$\therefore \quad$ stationary point is $\left(\dfrac{a}{\sqrt{3}}, \dfrac{b}{\sqrt{3}}, \dfrac{c}{\sqrt{3}}\right)$

At $\left(\dfrac{a}{\sqrt{3}}, \dfrac{b}{\sqrt{3}}, \dfrac{c}{\sqrt{3}}\right)$,

$$d^2F = \Sigma F_{xx}(dx)^2 + 2\Sigma F_{yz}\,dy\,dz = 2\lambda\left[\frac{(dx)^2}{a^2} + \frac{(dy)^2}{b^2} + \frac{(dz)^2}{c^2}\right] + 16\Sigma\, a\,dy\,dz$$

$$\therefore \quad d^2F = -\frac{8abc}{\sqrt{3}}\left[\frac{(dx)^2}{a^2} + \frac{(dy)^2}{b^2} + \frac{(dz)^2}{c^2}\right] + \frac{16}{\sqrt{3}}(a\,dy\,dz + b\,dz\,dx + c\,dx\,dy) \qquad \ldots(6)$$

Diff. (2), we get,

$$2\left(\frac{x}{a^2}dx + \frac{y}{b^2}dy + \frac{z}{c^2}dz\right) = 0$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Putting $x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$, we get

$$\frac{dx}{a} + \frac{dy}{b} + \frac{dz}{c} = 0$$

$$\therefore \quad dz = -c\left(\frac{dx}{a} + \frac{dy}{b}\right)$$

$\therefore$ from (6), we get,

$$d^2 F = -\frac{8abc}{\sqrt{3}}\left[\frac{(dx)^2}{a^2} + \frac{(dy)^2}{b^2} + \left(\frac{dx}{a} + \frac{dy}{b}\right)^2\right]$$

$$+ \frac{16}{\sqrt{3}}\left[-c\,a\,dy\left(\frac{dx}{a} + \frac{dy}{b}\right) - b\,c\,dx\left(\frac{dx}{a} + \frac{dy}{b}\right) + c\,dx\,dy\right]$$

$$= -\frac{32abc}{\sqrt{3}}\left[\left(\frac{dx}{a}\right)^2 + \frac{dx\,dy}{ab} + \left(\frac{dy}{b}\right)^2\right] = -\frac{32abc}{\sqrt{3}}\left[\left(\frac{dx}{a} + \frac{1}{2}\frac{dy}{b}\right)^2 + \frac{3}{4}\left(\frac{dy}{b}\right)^2\right]$$

which is negative definite form

$\therefore$ V is max. at $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$

and max. value $= 8\frac{a}{\sqrt{3}} \cdot \frac{b}{\sqrt{3}} \cdot \frac{c}{\sqrt{3}} = \frac{8abc}{3\sqrt{3}}$

Note. Show that the volume of the largest rectangular parallelopiped that can be inscribed in the sphere $x^2 + y^2 + z^2 = a^2$ is $\frac{8a^3}{3\sqrt{3}}$.

Note. Put $b = a$, $c = a$ in above solution.

Example 5. Find the shortest distance from the origin to the surface $l\,x + m\,y + n\,z = p$.

Sol. Let (x, y, z) be any point on the surface

$$l\,x + m\,y + n\,z = p \qquad ...(1)$$

Let $r^2 = x^2 + y^2 + z^2$ ...(2)

where r is the distance of (0, 0, 0) from (x, y, z)

Let $F(x, y, z) = x^2 + y^2 + z^2 + \lambda\,(l\,x + m\,y + n\,z - p)$

For extreme values

$$F_x = 2\,x + \lambda\,l = 0 \qquad ...(3)$$

$$F_y = 2\,y + \lambda\,m = 0 \qquad ...(4)$$

$$F_z = 2\,z + \lambda\,n = 0 \qquad ...(5)$$

From (3), (4), (5), we get,

$$x = -\frac{\lambda l}{2}, \; y = -\frac{\lambda m}{2}, \; z = -\frac{\lambda n}{2}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Putting these values of x, y, z in (1), we get

$$-\frac{\lambda}{2}l^2-\frac{\lambda}{2}m^2-\frac{\lambda}{2}n^2=p$$

$$\therefore \quad \lambda=-\frac{2p}{l^2+m^2+n^2}$$

$$\therefore \quad x=\frac{pl}{l^2+m^2+n^2}, \quad y=\frac{pm}{l^2+m^2+n^2}, \quad z=\frac{pn}{l^2+m^2+n^2}$$

Now $d^2F = \Sigma F_{xx}(dx)^2 + 2\Sigma F_{yz}\,dy\,dz$

$$= 2[(dx)^2+(dy)^2+(dz)^2]+0$$

$$= 2[(dx)^2+(dy)^2+(dz)^2]$$

which is positive definite

$\therefore$ r is minimum

$\therefore$ minimum value of r

$$=\left[\frac{p^2l^2}{(l^2+n^2+n^2)^2}+\frac{p^2m^2}{(l^2+m^2+n^2)^2}+\frac{p^2n^2}{(l^2+m^2+n^2)^2}\right]^{\frac{1}{2}}=\frac{p}{\sqrt{l^2+m^2+n^2}}.$$

Example 6. Find the extreme values of r^2 where $r^2=\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}$, subject to the conditions $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, $lx+my+nz=0$.

Sol. We want to find the extreme values of

$$r^2=\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4} \qquad \ldots(1)$$

subject to the conditions

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \ldots(2)$$

and $lx+my+nz=0$ ...(3)

Consider the function

$$F(x,y,z)=\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)+\lambda_1\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)+\lambda_2(lx+my+nz)$$

where λ_1, λ_2 are Lagrange's multipliers to be determined.

For extreme values,

$$F_x=\frac{2x}{a^4}+\frac{2x}{a^2}\lambda_1+l\lambda_2=0 \qquad \ldots(4)$$

$$F_y=\frac{2y}{b^4}+\frac{2y}{b^2}\lambda_1+m\lambda_2=0 \qquad \ldots(5)$$

$$F_z=\frac{2z}{b^4}+\frac{2z}{b^2}\lambda_1+n\lambda_2=0 \qquad \ldots(6)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Multiplying (4) by x, (5) by y, (6) by z and adding, we get,

$$2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)+2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\lambda_1+(lx+my+nz)\lambda_2=0$$

or $\quad 2r^2+2(1)\lambda_1+0(\lambda_2)=0 \qquad [\because \text{ of } (1), (2), (3)]$

or $\quad \lambda_1=-r^2$

Putting this value of λ_1 in (4), (5), (6), we get,

$$\frac{2x}{a^2}\left(\frac{1}{a^2}-r^2\right)=-l\lambda_2 \quad \Rightarrow \quad x=-\frac{a^4 l\lambda_2}{2(1-a^2r^2)}$$

$$\frac{2y}{b^2}\left(\frac{1}{b^2}-r^2\right)=-m\lambda_2 \quad \Rightarrow \quad y=-\frac{b^4 m\lambda_2}{2(1-b^2r^2)}$$

$$\frac{2z}{c^2}\left(\frac{1}{c^2}-r^2\right)=-n\lambda_2 \quad \Rightarrow \quad z=-\frac{c^4 n\lambda_2}{2(1-c^2r^2)}$$

Substituting these values of x, y and z in (3), we get,

$$-\frac{a^4l^2\lambda_2}{2(1-a^2r^2)}-\frac{b^4m^2\lambda_2}{2(1-b^2r^2)}-\frac{c^4n^2\lambda_2}{2(1-c^2r^2)}=0$$

or $$\frac{a^4l^2}{1-a^2r^2}+\frac{b^4m^2}{1-b^2r^2}+\frac{c^4n^2}{1-c^2r^2}=0$$

This equation gives extreme values of r^2.

Example 7. Find the maximum and minimum values of $x^2+y^2+z^2$ subject to the constraints

$$\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1 \text{ and } z=x+y$$

Sol. We want to find the extreme values of

$$r^2=x^2+y^2+z^2 \qquad ...(1)$$

subject to the conditions

$$\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1=0 \qquad ...(2)$$

and $x+y-z=0 \qquad ...(3)$

Consider the function,

$$F(x,y,z)=x^2+y^2+z^2+\lambda_1\left(\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1\right)+\lambda_2(x+y-z)$$

where λ_1, λ_2 are Lagrange's multipliers to be determined

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Putting these values of x, y, z in (1), we get

$$-\frac{\lambda}{2}l^2-\frac{\lambda}{2}m^2-\frac{\lambda}{2}n^2=p$$

$$\therefore\quad \lambda=-\frac{2p}{l^2+m^2+n^2}$$

$$\therefore\quad x=\frac{pl}{l^2+m^2+n^2},\ y=\frac{pm}{l^2+m^2+n^2},\ z=\frac{pn}{l^2+m^2+n^2}$$

Now $d^2F = \Sigma F_{xx}(dx)^2+2\Sigma F_{yz}\,dy\,dz$

$$= 2[(dx)^2+(dy)^2+(dz)^2]+0$$

$$= 2[(dx)^2+(dy)^2+(dz)^2]$$

which is positive definite

$\therefore$ r is minimum

$\therefore$ minimum value of r

$$=\left[\frac{p^2l^2}{(l^2+n^2+n^2)^2}+\frac{p^2m^2}{(l^2+m^2+n^2)^2}+\frac{p^2n^2}{(l^2+m^2+n^2)^2}\right]^{\frac{1}{2}}=\frac{p}{\sqrt{l^2+m^2+n^2}}.$$

Example 6. Find the extreme values of r^2 where $r^2=\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}$, subject to the conditions $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ lx+my+nz=0.$

Sol. We want to find the extreme values of

$$r^2=\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4} \qquad ...(1)$$

subject to the conditions

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad ...(2)$$

and $lx+my+nz=0$ ...(3)

Consider the function

$$F(x,y,z)=\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)+\lambda_1\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)+\lambda_2(lx+my+nz)$$

where λ_1, λ_2 are Lagrange's multipliers to be determined.

For extreme values,

$$F_x=\frac{2x}{a^4}+\frac{2x}{a^2}\lambda_1+l\lambda_2=0 \qquad ...(4)$$

$$F_y=\frac{2y}{b^4}+\frac{2y}{b^2}\lambda_1+m\lambda_2=0 \qquad ...(5)$$

$$F_z=\frac{2z}{b^4}+\frac{2z}{b^2}\lambda_1+n\lambda_2=0 \qquad ...(6)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Multiplying (4) by x, (5) by y, (6) by z and adding, we get,

$$2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)+2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\lambda_1+(l\,x+m\,y+n\,z)\,\lambda_2=0$$

or $\quad 2\,r^2+2\,(1)\,\lambda_1+0\,(\lambda_2)=0 \qquad [\because$ of (1), (2), (3)]

or $\quad \lambda_1=-r^2$

Putting this value of λ_1 in (4), (5), (6), we get,

$$\frac{2\,x}{a^2}\left(\frac{1}{a^2}-r^2\right)=-l\,\lambda_2 \quad\Rightarrow\quad x=-\frac{a^4\,l\,\lambda_2}{2\,(1-a^2\,r^2)}$$

$$\frac{2\,y}{b^2}\left(\frac{1}{b^2}-r^2\right)=-m\,\lambda_2 \quad\Rightarrow\quad y=-\frac{b^4\,m\,\lambda_2}{2\,(1-b^2r^2)}$$

$$\frac{2\,z}{c^2}\left(\frac{1}{c^2}-r^2\right)=-n\,\lambda_2 \quad\Rightarrow\quad z=-\frac{c^4\,n\,\lambda_2}{2\,(1-c^2r^2)}$$

Substituting these values of x, y and z in (3), we get,

$$-\frac{a^4l^2\lambda_2}{2\,(1-a^2\,r^2)}-\frac{b^4m^2\lambda_2}{2\,(1-b^2r^2)}-\frac{c^4n^2\lambda_2}{2\,(1-c^2r^2)}=0$$

or
$$\frac{a^4\,l^2}{1-a^2r^2}+\frac{b^4\,m^2}{1-b^2r^2}+\frac{c^4\,n^2}{1-c^2r^2}=0$$

This equation gives extreme values of r^2.

Example 7. Find the maximum and minimum values of $x^2+y^2+z^2$ subject to the constraints

$$\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1 \text{ and } z=x+y$$

Sol. We want to find the extreme values of

$$r^2=x^2+y^2+z^2 \qquad ...(1)$$

subject to the conditions

$$\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1=0 \qquad ...(2)$$

and $\quad x+y-z=0 \qquad ...(3)$

Consider the function,

$$F\,(x,y,z)=x^2+y^2+z^2+\lambda_1\left(\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1\right)+\lambda_2\,(x+y-z)$$

where λ_1, λ_2 are Lagrange's multipliers to be determined

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

For extreme values,

$$F_x = 2x + \frac{\lambda_1 x}{2} + \lambda_2 = 0 \qquad ...(4)$$

$$F_y = 2y + \frac{2\lambda_1 y}{5} + \lambda_2 = 0 \qquad ...(5)$$

$$F_z = 2z + \frac{2\lambda_1 z}{25} - \lambda_2 = 0 \qquad ...(6)$$

Multiplying (4) by x, (5) by y, (6) by z and adding, we get,

$$2(x^2 + y^2 + z^2) + \lambda_1 \left(\frac{x^2}{2} + \frac{2y^2}{5} + \frac{2z^2}{25}\right) + \lambda_2 (x + y - z) = 0$$

or $2r^2 + \lambda_1 (2) + \lambda_2 (0) = 0$ [$\because$ of (4), (5), (6)]

or $r^2 + \lambda_1 = 0 \quad \Rightarrow \quad \lambda_1 = -r^2$

From (4), (5) and (6), we get,

$$x = -\frac{2\lambda_2}{\lambda_1 + 4}, \; y = -\frac{5\lambda_2}{2\lambda_1 + 10}, \; z = \frac{25\lambda_2}{2\lambda_1 + 50}$$

$$\therefore \quad x = \frac{2\lambda_2}{r^2 - 4}, \; y = \frac{5\lambda_2}{2r^2 - 10}, \; z = -\frac{25\lambda_2}{2r^2 - 50}$$

Putting these values of x, y, z in (3), we get,

$$\frac{-2\lambda_2}{-r^2 + 4} - \frac{5\lambda_2}{-2r^2 + 10} - \frac{25\lambda_2}{-2r^2 + 50} = 0$$

$$\Rightarrow \quad \frac{2}{-r^2 + 4} + \frac{5}{-2r^2 + 10} + \frac{25}{-2r^2 + 50} = 0$$

[$\because \lambda_2 \neq 0$ as $\lambda_2 = 0$ gives $x = y = z = 0$]

$$\Rightarrow \quad 2(-2r^2 + 10)(-2r^2 + 50) + 5(-r^2 + 4)(-2r^2 + 50) + 25(-r^2 + 4)(-2r^2 + 10) = 0$$

$$\Rightarrow \quad 8r^4 - 240r^2 + 1000 + 10r^4 - 290r^2 + 1000 + 50r^4 - 450r^2 + 1000 = 0$$

$$\Rightarrow \quad 68r^4 - 980r^2 + 3000 = 0$$

$$\Rightarrow \quad 17r^4 - 245r^2 + 750 = 0$$

$$\therefore \quad r^2 = \frac{245 \pm \sqrt{(245)^2 - 4(17)(750)}}{34} = \frac{245 \pm \sqrt{60025 - 51000}}{34} = \frac{245 \pm \sqrt{9025}}{34}$$

$$= \frac{245 \pm 95}{34} = \frac{340}{34}, \frac{150}{34} = 10, \frac{75}{17}$$

$$\therefore \quad \lambda_1 = -10, \; -\frac{75}{17}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Take $\lambda_1 = -r^2 = -10$

$$\therefore \quad x = \frac{2\lambda_2}{10-4} = \frac{2\lambda_2}{6} = \frac{\lambda_2}{3}, \quad y = \frac{5\lambda_2}{20-10} = \frac{5\lambda_2}{10} = \frac{\lambda_2}{2}$$

$$z = -\frac{25\lambda_2}{20-50} = \frac{25\lambda_2}{30} = \frac{5\lambda_2}{6}$$

Putting values of x, y, z in (2), we get,

$$\frac{1}{36}\lambda_2^2 + \frac{1}{20}\lambda_2^2 + \frac{1}{36}\lambda_2^2 = 1$$

or $\lambda_2^2 = \dfrac{180}{19} \qquad \Rightarrow \lambda_2 = \pm 6\sqrt{\dfrac{5}{19}}$

$\therefore$ two stationary points are

$$\left(2\sqrt{\frac{5}{19}}, 3\sqrt{\frac{5}{19}}, 5\sqrt{\frac{5}{19}}\right) \text{ and } \left(-2\sqrt{\frac{5}{19}}, -3\sqrt{\frac{5}{19}}, -5\sqrt{\frac{5}{19}}\right)$$

$$\therefore \quad r^2 = \frac{20}{19} + \frac{45}{19} + \frac{125}{19} = \frac{190}{19} = 10$$

Take $\lambda_1 = -r^2 = -\dfrac{75}{17}$

$$\therefore \quad x = \frac{2\lambda_2}{\frac{75}{17}-4} = \frac{34\lambda_2}{7}$$

$$y = \frac{5\lambda_2}{\frac{150}{17}-10} = \frac{85\lambda_2}{-20} = -\frac{17}{4}\lambda_2$$

$$z = -\frac{25\lambda_2}{\frac{150}{17}-50} = -\frac{25\times 17\lambda_2}{-700} = \frac{17}{28}\lambda_2$$

Putting these values of x, y, z in (2), we get

$$\frac{\left(\frac{34\lambda_2}{7}\right)^2}{4} + \frac{\left(-\frac{17}{4}\lambda_2\right)^2}{5} + \frac{\left(\frac{17\lambda_2}{28}\right)^2}{25} = 1$$

$$\frac{289}{49}\lambda_2^2 + \frac{289\lambda_2^2}{80} + \frac{289}{19600}\lambda_2^2 = 1$$

$$\Rightarrow \quad \lambda_2^2 = \frac{19600}{289\times 646} \qquad \Rightarrow \lambda_2 = \pm\frac{140}{17\sqrt{646}}$$

$\therefore$ two stationary points are

$$\left(\frac{40}{\sqrt{646}}, -\frac{35}{\sqrt{646}}, \frac{5}{\sqrt{646}}\right) \text{ and } \left(-\frac{40}{\sqrt{646}}, \frac{35}{\sqrt{646}}, \frac{-5}{\sqrt{646}}\right)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College of Education Bantalab Jammu

$$\therefore \quad r^2 = \frac{1600}{646} + \frac{1225}{646} + \frac{25}{646} = \frac{2850}{646} = \frac{75}{17}$$

$\therefore$ maxi. value = 10, mini. value = $\frac{75}{17}$.

EXERCISE 14 (c)

1. Find the minimum value of the function $x^2 + y^2 + z^2$ subject to the condition $x + y + z = 3a$.
2. Find the minimum value of $x^2 + y^2 + z^2$ when $yz + zx + xy = 3a^2$.
3. Find the maximum or minimum value of $x^2 + y^2 + z^2$ where $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$.
4. Find the volume of the largest rectangular parallelopiped which can be inscribed in the

   (*i*) ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$

   (*ii*) sphere $x^2 + y^2 + z^2 = 3$
5. Prove that the volume of the greatest rectangular parallelopiped that can be inscribed in the sphere $x^2 + y^2 + z^2 = a^2$ is $\frac{8a^3}{3\sqrt{3}}$.
6. Find the dimensions of a rectangular parallelopiped of given surface area S and maximum volume.
7. Find the maximum value of function $f(x, y, z) = xyz$ where x, y, z are related by $xy + yz + zx = a$, constant where x, y, z are non-zero and $a \geq 0$.
8. Find the shortest distance from the origin to the hyperbola $x^2 + 8xy + 7y^2 = 225$.
9. Which point on the sphere $x^2 + y^2 + z^2 = 1$ is at the maximum distance from the point (2, 1, 3) ?
10. Using Lagrange's method of multipliers, find the point on the plane $2x - 3y + 6z = 49$ nearest to the origin in $\mathbf{R}^3$.
11. Find the point on the plane $2x + 3y - z = 5$ which is nearest the origin in $\mathbf{R}^3$.
12. If $u = a^3x^2 + b^3y^2 + c^3z^2$ where $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$, show that a stationary value is given by $ax = by = cz$ and this gives a maximum or minimum if $abc(a + b + c) > 0$.
13. Find the maximum and minimum values of r^2 where $r^2 = a^2x^2 + b^2y^2 + c^2z^2$, subject to the constraints $x^2 + y^2 + z^2 = 1$, $lx + my + nz = 0$.
14. Find the maximum and minimum radii vectors of the section of the surface $(x^2 + y^2 + z^2)^2 = l^2x^2 + m^2y^2 + n^2z^2$ made by the plane $ax + by + cz = 0$.
15. Find the lengths of the axes of the section of ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by the plane $lx + my + nz = 0$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

16. Find the maximum and minimum radii vectors of the section of the surface $a x^2 + b y^2 + c z^2 = 1$ by the plane $l x + m y + n z = 0$.

OR

Find the extreme value of $x^2 + y^2 + z^2$ subject to the conditions $a x^2 + b y^2 + c z^2 = 1$ and $l x + m y + x z = 0$.

ANSWERS

1. $3 a^2$

2. $3 a^2$

3. Roots of the equation $\begin{vmatrix} a-\frac{1}{u} & h & g \\ h & b-\frac{1}{u} & f \\ g & f & c-\frac{1}{u} \end{vmatrix} = 0$

give maximum or minimum value.

4. (*i*) $\frac{64}{\sqrt{3}}$ cubic units (*ii*) 8 cubic units

6. Each side $= \sqrt{\frac{S}{6}}$

7. $\left(\frac{a}{3}\right)^{\frac{3}{2}}$

8. 5

9. $\left(-\frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\right)$

10. $(2, -3, 6)$

11. $\left(\frac{5}{7}, \frac{15}{14}, -\frac{5}{14}\right)$

13. Values of r^2 given by $\frac{l^2}{a^2-r^2} + \frac{m^2}{b^2-r^2} + \frac{n^2}{c^2-r^2} = 0$

give maximum or minimum values.

14. Values of r^2 given by $\frac{a^2}{r^2-l^2} + \frac{b^2}{r^2-m^2} + \frac{c^2}{r^2-n^2} = 0$

give maximum or minimum values.

15. Values of r^2 given by $\frac{a^2l^2}{a^2-r^2} + \frac{b^2m^2}{b^2-r^2} + \frac{c^2n^2}{c^2-r^2} = 0$

give maximum or minimum values.

16. Values of r^2 given by $\frac{l^2}{1-ar^2} + \frac{m^2}{1-br^2} + \frac{n^2}{1-cr^2} = 0$

give maximum or minimum values

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

15

GENERAL THEOREMS

Art-1. Introduction

By now, the student is familiar to distinguish between theorems applicable to a class of functions like Trigonometric, Logarithmic functions etc. The theorems applicable to a class of functions are known as **General Theorems.** Out of these general theorems, which play an important role in Mathematics, the most fundamental is **Rolle's Theorem.** This theorem was given by the French Mathematician Michal Rolle in 1691.

Art-2. Rolle's Theorem

Statement : If a function f is

(*i*) continuous in the closed interval $[a, b]$

(*ii*) derivable in the open interval (a, b)

(*iii*) $f(a) = f(b)$

then there exists at least one real number $c \in (a, b)$ such that $f'(c) = 0$.

Proof. Case I. If f is a constant function in $[a, b]$,

then $f'(x) = 0 \ \forall \ x \in [a, b] \quad \Rightarrow \quad f'(c) = 0 \ \forall \ c \in (a, b)$.

$\therefore$ the result is true in this case.

Case II. If f is not a constant function throughout $[a, b]$, then as f is continuous in $[a, b]$, it is bounded and attains its *l.u.b.* and *g.l.b.* in $[a, b]$.

$\because$ f is not a constant function

$\therefore$ at least one of the these bounds (*l.u.b.* and *g.l.b.*) must be different from the equal values $f(a)$, $f(b)$ and let this bound be attained at $x = c$ (say) where c is different from a and b *i.e.*, $c \in (a, b)$.

Without any loss of generality, we assume that *l.u.b.* of f is attained at $x = c$.

$\therefore$ $f(c + h) \leq f(c)$, where $c + h$ is any point in the *nbd.* of c, h may be positive or negative.

If $h > 0$, then $\dfrac{f(c+h) - f(c)}{h} \leq 0 \quad \Rightarrow \quad \underset{h \to 0+}{\text{Lt}} \dfrac{f(c+h) - f(c)}{h} \leq 0$

$\Rightarrow \ \text{R}f'(c) \leq 0$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Again if $h < 0$, then $\dfrac{f(c+h)-f(c)}{h} \geq 0 \quad \Rightarrow \quad \underset{h \to 0-}{\text{Lt}} \dfrac{f(c+h)-f(c)}{h} \geq 0$

$\Rightarrow \quad \text{L}f'(c) \geq 0$

$\because \quad f(x)$ is derivable in (a, b)

$\therefore \quad f'(x)$ exists $\forall\ x \in (a, b) \qquad \Rightarrow \quad f'(c)$ exists

$\Rightarrow \quad \text{R}f'(c) = \text{L}f'(c) = f'(c)$

$\Rightarrow \quad f'(c) = 0 \qquad [\because 0$ is the common value of $\text{L}f'(c)$ and $\text{R}f'(c)]$

$\Rightarrow \quad f'(c) = 0$ for some $c \in (a, b)$

$\therefore$ the theorem is proved.

Art-3. Geometrical Interpretation of Rolle's Theorem

Let AB be the graph of function $y = f(x)$ such that the points A and B of the graph correspond to the numbers a and b of the interval $[a, b]$.

$\because \quad f(x)$ is continuous in the interval $[a, b]$

$\therefore$ its graph is a continuous curve between A and B.

Again as $f(x)$ is derivable in the open interval (a, b), therefore, graph of $f(x)$ has a unique tangent at every point between A and B.

$\because \quad f(a) = f(b)$

$\therefore$ AM = BN *i.e.*, ordinates of A and B are equal.

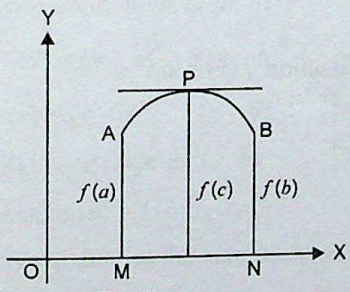


Fig. (i)

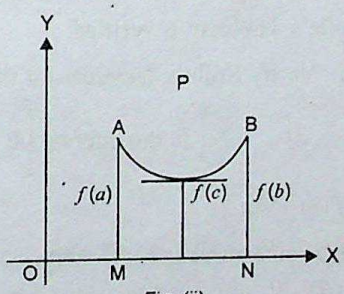


Fig. (ii)

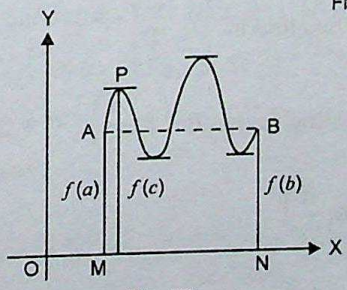


Fig. (iii)

From the three figures drawn above, it is clear that there is at least one point P on the curve between A and B, the tangent at which is parallel to X-axis.

$\therefore$ slope of tangent at P = 0

$\therefore \quad f'(c) = 0$, where c is abscissa of P.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

ILLUSTRATIVE EXAMPLES

Example 1. Verify Rolle's Theorem for the function $f(x) = x^2 - 4x + 3$ in the interval $1 \le x \le 3$.

Sol. Here $f(x) = x^2 - 4x + 3$

It is a polynomial in x.

(*i*) Since every polynomial is a continuous function for every real value of x.

$\therefore$ f is continuous in $1 \le x \le 3$.

(*ii*) $f'(x) = 2x - 4$, which exists in $1 < x < 3$

$\therefore$ f is derivable in $1 < x < 3$.

(*iii*) $f(1) = (1)^2 - 4 \times 1 + 3 = 1 - 4 + 3 = 0$

$f(3) = (3)^2 - 4 \times 3 + 3 = 9 - 12 + 3 = 0$

$\therefore$ $f(1) = f(3)$

$\therefore$ f satisfies all the conditions of Rolle's Theorem.

$\therefore$ there must exist at least one value c of x such that $f'(c) = 0$, where $1 < c < 3$.

Now $f'(c) = 0$ gives us $2c - 4 = 0$ or $c = 2$

Now $c = 2$ lies in the open interval $(1, 3)$

$\therefore$ Rolle's Theorem is verified.

Example 2. Verify Rolle's Theorem for the function $f(x) = \sin 2x$ or $\cos 2\left(x - \frac{\pi}{4}\right)$ in the interval $\left[0, \frac{\pi}{2}\right]$.

Sol. Here $f(x) = \sin 2x$

(*i*) Since sine curve is continuous

$\therefore$ f is a continuous function in $\left[0, \frac{\pi}{2}\right]$

(*ii*) $f'(x) = 2\cos 2x$, which exists in $\left(0, \frac{\pi}{2}\right)$

$\therefore$ f is derivable in $\left(0, \frac{\pi}{2}\right)$

(*iii*) $f(0) = \sin 0 = 0$

$\therefore$ $f\left(\frac{\pi}{2}\right) = \sin \pi = 0$

$\therefore$ $f(0) = f\left(\frac{\pi}{2}\right)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$\therefore$ f satisfies all the conditions of Rolle's Theorem

$\therefore$ there must exists at least one value c of x such that

$$f'(c) = 0 \text{ where } c \in \left(0, \frac{\pi}{2}\right).$$

Now $f'(c) = 2\cos 2c$

$\therefore$ $f'(c) = 0$ gives us $2\cos 2c = 0$, *i.e.*, $\cos 2c = 0$ *i.e.*, $2c = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

But $2c = \frac{\pi}{2}$ or $c = \frac{\pi}{4}$ lies in $\left(0, \frac{\pi}{2}\right)$

$\therefore$ Rolle's Theorem is verified.

Example 3. Show that $f(x) = (x-a)^m (x-b)^n$, m, n being the integers, satisfies Rolle's Theorem in $[a, b]$. Find the value of c.

Sol. $f(x) = (x-a)^m (x-b)^n$

$\because$ m, n are + ve integers,

$\therefore$ after expansion by Binomial theorem, $(x-a)^m$, $(x-b)^n$ are polynomials in x.

$\therefore$ $f(x) = (x-a)^m (x-b)^n$ is also a polynomial in x.

(*i*) Since every polynomial in x is a continuous function of x for every value of x,

$\therefore$ $f(x)$ is continuous in $[a, b]$

(*ii*) $f'(x) = (x-a)^m . n(x-b)^{n-1} + (x-b)^n . m(x-a)^{m-1}$.

$\therefore$ $f'(x) = (x-a)^{m-1}(x-b)^{n-1}[m(x-b) + n(x-a)]$ which exists in (a, b)

$\therefore$ $f(x)$ is derivable in (a, b)

(*iii*) Also $f(a) = f(b) = 0$

$\therefore$ $f(x)$ satisfies all the conditions of Rolle's Theorem.

$\therefore$ there exists at least one value c of x in (a, b) such that $f'(c) = 0$

$\therefore$ $(c-a)^{m-1}(c-b)^{n-1}[m(c-b) + n(c-a)] = 0$

$\therefore$ $m(c-b) + n(c-a) \neq 0$ $\quad [\because c \neq a, c \neq b]$

$\Rightarrow$ $(m+n)c = mb + na$

$c = \frac{mb+na}{m+n} \in (a, b)$ as c is the point which divides (a, b) internally in the ratio $m : n$.

Example 4. Discuss the applicability of Rolle's Theorem to $f(x) = |x|$ in the interval $[-1, 1]$.

Sol. Here $f(x) = |x|$

We shall show that f is not differentiable at $x = 0$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\text{L.H.D.} = \underset{x \to 0-}{\text{Lt}} \frac{f(x) - f(0)}{x - 0} = \underset{x \to 0-}{\text{Lt}} \frac{|x| - |0|}{x - 0} = \underset{x \to 0-}{\text{Lt}} \frac{|x|}{x} = \underset{h \to 0}{\text{Lt}} \frac{|0 - h|}{0 - h}$$

[Put $x = 0 - h, h > 0$ so that $h \to 0$ as $x \to 0-$]

$$= \underset{h \to 0}{\text{Lt}} \frac{|-h|}{-h} = \underset{h \to 0}{\text{Lt}} \frac{h}{-h} = -1.$$

$$\text{R.H.D.} = \underset{x \to 0+}{\text{Lt}} \frac{f(x) - f(0)}{x - 0} = \underset{x \to 0+}{\text{Lt}} \frac{|x|}{x} = \underset{h \to 0}{\text{Lt}} \frac{|0 + h|}{0 + h}$$

[Put $x = 0 + h, h > 0$ so that $h \to 0$ as $x \to 0+$]

$$= \underset{h \to 0}{\text{Lt}} \frac{|h|}{h} = \underset{h \to 0}{\text{Lt}} \frac{h}{h} = 1$$

$\therefore$ L.H.D. $\neq$ R.H.D.

$\therefore$ $f(x) = |x|$ is not differentiable at $x = 0 \in (-1, 1)$

$\therefore$ Rolle's theorem cannot be applied to $f(x) = [x]$ in $[-1, 1]$.

Example 5. Apply Rolle's Theorem to locate the zeros of $f'(x)$ where

$$f(x) = 2(x-1)(x-4)(2x-9)(x-7).$$

Sol. Here $f(x) = 2(x-1)(x-4)(2x-9)(x-7)$

(*i*) $\therefore$ $f(x)$ is a polynomial in x

$\therefore$ f is continuous $\forall\ x \in \mathbf{R}$.

(*ii*) $\because$ $f(x)$ is a polynomial in x, $\therefore$ $f(x)$ is derivable for all $\forall\ x \in \mathbf{R}$

(*iii*) $f(1) = f(4) = f\left(\frac{9}{2}\right) = f(7) = 0$

$\therefore$ f satisfies all the conditions of Rolle's Theorem in each of the intervals $(1, 4)$, $\left(4, \frac{9}{2}\right), \left(\frac{9}{2}, 7\right)$.

$\therefore$ there exists at least one real number x in each of the intervals $(1, 4)$, $\left(4, \frac{9}{2}\right), \left(\frac{9}{2}, 7\right)$ where $f'(x) = 0$

Now $f(x)$ is of degree 4 and consequently $f'(x)$ is of degree 3.

Also $f'(x)$ has at least one real zero in $(1, 4)$, $\left(4, \frac{9}{2}\right), \left(\frac{9}{2}, 7\right)$.

$\therefore$ $f'(x)$ has exactly one zero in $(1, 4)$, $\left(4, \frac{9}{2}\right), \left(\frac{9}{2}, 7\right)$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 6. Let f, g, h be three continuous functions in the closed interval $[a, b]$ and differentiable in the open interval (a, b). Show that there exists a number $c \in (a, b)$ such that $\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$

Sol. Consider a function

$$\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

Here $\phi(x)$, when expanded, is of the form $A f(x) + B g(x) + C h(x)$ where A, B, C are all constants.

$\because$ f, g, h are continuous in $[a,. b]$ and derivable in (a, b)

$\therefore$ ϕ is continuous in $[a, b]$ and derivable in (a, b).

Also $\phi(a) = \phi(b) = 0$

[$\because$ two rows of $\phi(x)$ become identical for both $x = a$ and $x = b$]

$\therefore$ ϕ satisfies all the conditions of Rolle's Theorem.

$\therefore$ there exists a real number $c \in (a, b)$ such that

$$\phi'(c) = 0 \qquad \ldots(1)$$

Now $\phi'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$

$\therefore$ from (1), we get,

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0, \text{ where } c \in (a, b).$$

Hence the result.

EXERCISE 15 (a)

1. Verify Rolle's Theorem for the following functions :

(*i*) $f(x) = x^2 - 5x + 4$ in $1 \le x \le 4$

(*ii*) $f(x) = x^2$ in $[-1, 1]$

(*iii*) $f(x) = 2(x+1)(x-2)$ in $[-1, 2]$

(*iv*) $f(x) = \cos 2x$ in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

(*v*) $f(x) = e^x \cos x$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(*vi*) $f(x) = \sqrt{1 - x^2}$ in $[-1, 1]$

(*vii*) $f(x) = |9 - x^2|$ in $[-3, 3]$

(*viii*) $f(x) = x(x+3)\, e^{-\frac{x}{2}}$ in $[-3, 0]$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

2. Discuss the applicability of Rolle's Theorem for the following functions :

(*i*) $f(x) = 2 + (x-1)^{\frac{2}{3}}$ in $[0, 2]$ (*ii*) $f(x) = x^2$ in $[1, 2]$

(J.U. 2016)

(*iii*) $f(x) = \begin{cases} \dfrac{x^2 - 5x + 4}{x - 1}, & x \neq 1 \\ 0, & x = 1 \end{cases}$ (*iv*) $f(x) = 1 + |x - 2|$ in $[0, 4]$.

in $[1, 4]$

3. (*a*) If $f(x) = x^4 - 2x^3 + 2x^2 - x$, then prove by Rolle's Theorem that the equation $4x^3 - 6x^2 + 4x - 1 = 0$ has at least one real root in $(0, 1)$.

(*b*) Apply Rolle's Theorem to find position of real zeros of $f'(x)$ where $f(x) = x(x-1)(x-2)(x-3)$.

4. Determine by Rolle's Theorem, the number of zeros of the derivative of the following functions :

(*i*) $f(x) = (x^2 - 1)(x^2 - 4), -2 \leq x \leq 2$

(*ii*) $f(x) = \sin 2x, \ 0 \leq x \leq 2\pi$

5. By considering the function $(x-2)\log x$, show that the equation $x \log x = 2 - x$ is satisfied by at least one value of x between 1 and 2.

6. Let f and g be two differentiable functions in $[a, b]$, and $f(a) = g(a), f(b) = g(b)$; show that $f'(c) = g'(c)$ for some c in (a, b).

Hence or otherwise show that $f'(x) = g'(x)$ for at least one x in $(0, 1)$

where $f(x) = \dfrac{x}{1+x^2}$ and $g(x) = \dfrac{x^2}{1+x^2}$.

ANSWERS

2. (*i*) Not applicable (*ii*) Not applicable

(*iii*) Not applicable (*iv*) Not applicable

4. (*i*) Exactly one zero in $(-2, -1), (-1, 1), (1, 2)$

(*ii*) $\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$ in $(0, 2\pi)$

Art-4. Lagrange's Mean Value Theorem

Statement. If a function f is

(*i*) continuous in $[a, b]$

(*ii*) differentiable in (a, b),

then there exists at least one real number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Proof. Consider a function

$$\phi(x) = f(x) + \mathrm{A}\,x \qquad \ldots(1)$$

where A is a constant to be determined such that $\phi(a) = \phi(b)$

i.e., $f(a) + \mathrm{A}\,.\,a = f(b) + \mathrm{A}\,.\,b$

i.e., $\mathrm{A}(a - b) = f(b) - f(a)$

i.e., $$-\mathrm{A} = \frac{f(b) - f(a)}{b - a} \qquad \ldots(2)$$

(*i*) Now $f(x)$ is continuous in $[a, b]$

Also $\mathrm{A}\,x$, being polynomial in x, is continuous in $[a, b]$

$\therefore$ $\phi(x)$, being the sum of two continuous functions in $[a, b]$, is continuous in $[a, b]$.

(*ii*) $\phi(x)$, being the sum of two derivable functions $f(x)$ and $\mathrm{A}\,x$ in (a, b), is derivable in (a, b).

(*iii*) Also $\phi(a) = \phi(b)$

$\therefore$ $\phi(x)$ satisfies all the conditions of Rolle's Theorem.

$\therefore$ $\phi'(c) = 0$ where $a < c < b$

Now $\phi'(x) = f'(x) + \mathrm{A} \Rightarrow \phi'(c) = f'(c) + \mathrm{A}$

$\therefore$ $\phi'(c) = 0$ gives us $f'(c) + \mathrm{A} = 0$

$\therefore$ $$-\mathrm{A} = f(c) \qquad \ldots(3)$$

From (2) and (3), we get,

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Cor. Another Form :

Put $b = a + h\ (h > 0)$

$$\therefore \quad \frac{f(a+h) - f(a)}{a + h - a} = f'(c), \text{ where } a < c < a + h$$

$$\therefore \quad \frac{f(a+h) - f(a)}{h} = f'(a + \theta h),\ 0 < \theta < 1$$

$[\because$ for $0 < \theta < 1,\ a < a + \theta h < a + h]$

$$\therefore \quad f(a+h) - f(a) = h f'(a + \theta h)$$

or $$f(a+h) = f(a) + h f'(a + \theta h), \text{ where } 0 < \theta < 1$$

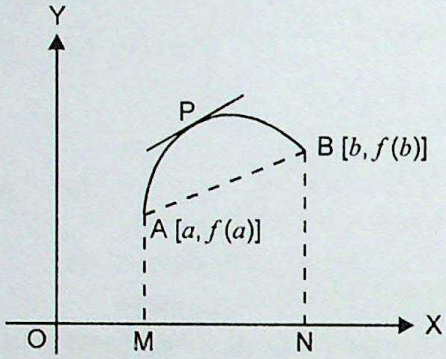
CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

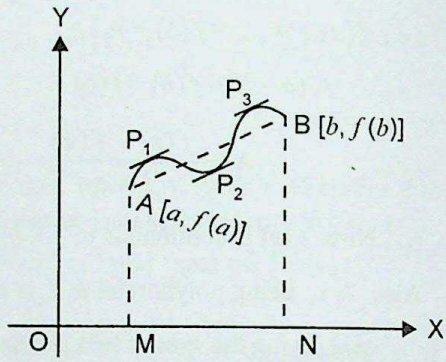
Gandhi Memorial College Of Education Bantalab Jammu

Art-5. Geometrical Interpretation

When $x = a,\ f(x) = f(a)$

When $x = b,\ f(x) = f(b)$





$\therefore$ $(a, f(a))$, $(b, f(b))$ are two points on the graph of $y = f(x)$. Let these points be A and B.

$\because$ $f(x)$ is continuous in $a \le x \le b$

$\therefore$ graph of $f(x)$ is continuous for $a \le x \le b$.

$\because$ $f(x)$ is derivable for $a < x < b$.

$\therefore$ tangent exists at each point of the graph for $a < x < b$.

$\therefore$ there exists at least one point P on graph between the points A and B such that the tangent at P is parallel to the chord AB. If c is abscissa of P, then

$$f'(c) = \text{slope of chord AB} = \frac{f(b) - f(a)}{b - a}$$

Note. Lagrange's Mean Value Theorem is also known as *first mean value theorem* or *mean value theorem* or *law of mean.*

Art-6. Applications of Mean Value Theorem

(I) If a functions f is

(*i*) continuous in the closed interval $[a, b]$

(*ii*) derivable in the open interval (a, b) and

(*iii*) $f'(x) = 0 \quad \forall\ x \in (a, b)$

then f is constant in $[a, b]$.

Proof. Let x_1, x_2 be any two real numbers in $[a, b]$ such that $x_1 < x_2$. Then f satisfies all the conditions of Lagrange's Mean Value Theorem in $[x_1, x_2]$.

$\therefore$ the exist at least one real number $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

But $f'(x) = 0 \ \forall \ x \in (a, b)$ so that $f'(c) = 0$

$$\Rightarrow \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0 \quad \Rightarrow \quad f(x_2) - f(x_1) = 0$$

$\Rightarrow \quad f(x_1) = f(x_2)$ for all $x_1, x_2 \in [a, b]$

$\Rightarrow \quad f(x)$ is constant in $[a, b]$.

(II) If $f(x)$, $g(x)$ are two functions such that

(*i*) $f(x)$, $g(x)$ are both continuous in $[a, b]$

(*ii*) $f(x)$, $g(x)$ are both derivable in (a, b) and

(*iii*) $f'(x) = g'(x) \quad \forall \ x \in (a, b)$

then $f(x)$ and $g(x)$ differ by a constant in $[a, b]$.

Proof. Consider $\phi(x) = f(x) - g(x)$

(*i*) $\phi(x)$ is continuous in $[a, b]$ as $f(x)$, $g(x)$ are both continuous in $[a, b]$.

(*ii*) $\phi(x)$ is derivable in (a, b) as $f(x)$, $g(x)$ are both derivable in (a, b)

(*iii*) Also $\phi'(x) = f'(x) - g'(x)$

$= 0 \qquad [\because f'(x) = g'(x) \ \forall \ x \in (a, b)]$

$\therefore \quad \phi'(x) = 0 \quad \forall \ x \in (a, b)$

$\therefore \quad \phi(x)$ is constant for all x in $[a, b]$

$\Rightarrow \quad f(x) - g(x)$ is constant $\forall \ x \in [a, b]$

$\Rightarrow \quad f(x)$ and $g(x)$ differ by a constant $\forall \ x \in [a, b]$

Note. On the lines of above two results, we can prove the following two results :

(1) If $f(x)$ is derivable in (a, b) such that $f'(x) = 0 \ \forall \ x \in (a, b)$, then $f(x)$ is a constant in (a, b).

(2) If $f(x)$ and $g(x)$ are differentiable functions in (a, b) such that $f'(x) = g'(x) \ \forall \ x \in (a, b)$ then $f(x)$ and $g(x)$ differ by a constant in (a, b).

ILLUSTRATIVE EXAMPLES

Example 1. Verify Lagrange's mean value theorem for the following functions :

$f(x) = \mathrm{A}x^2 + \mathrm{B}x + \mathrm{C}$ in $[a, b]$. Also find c of the theorem.

Sol. Here $f(x) = \mathrm{A}x^2 + \mathrm{B}x + \mathrm{C}$

$\therefore$ $f(x)$ is continuous in $[a, b]$ and derivable in (a, b).

$\therefore$ by Mean value theorem, there exists at least one $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow 2\mathrm{A}c + \mathrm{B} = \frac{(\mathrm{A}b^2 + \mathrm{B}b + \mathrm{C}) - (\mathrm{A}a^2 + \mathrm{B}a + \mathrm{C})}{b - a}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\Rightarrow \quad 2\,A\,c + B = \frac{A\,(b^2 - a^2) + B\,(b - a)}{b - a} \quad \Rightarrow \quad 2\,A\,c + B = A\,(b + a) + B$$

$$\Rightarrow \quad 2\,A\,c = A\,(b + a) \quad \Rightarrow \quad c = \frac{a + b}{2} \in (a, b).$$

Example 2. In Lagrange's Mean Value Theorem, $f(a + h) = f(a) + h\,f'(a + \theta\,h)$.

Show that if $f(x) = A\,x^2 + B\,x + C$, $A \neq 0$, then $\theta = \frac{1}{2}$

Sol. Here $f(x) = A\,x^2 + B\,x + C$

$\therefore$ $f(x)$ is continuous in $[a, a + h]$ and derivable in $(a, a + h)$

Also $f'(x) = 2\,A\,x + B$

Now in Lagrange's Mean Value Theorem,

$$f(a + h) = f(a) + h\,f'(a + \theta\,h)$$

$$\Rightarrow \quad A\,(a + h)^2 + B\,(a + h) + C = A\,a^2 + B\,a + C + h\,[2\,A\,(a + \theta\,h) + B]$$

$$\Rightarrow \quad A\,(a^2 + 2\,a\,h + h^2) + B\,(a + h) + C = A\,a^2 + B\,a + C + h\,[2\,A\,a + 2\,A\,\theta\,h + B]$$

$$\Rightarrow \quad A\,a^2 + 2\,A\,a\,h + A\,h^2 + B\,a + B\,h + C = A\,a^2 + B\,a + C + 2\,A\,a\,h + 2\,A\,\theta\,h^2 + B\,h$$

$$\Rightarrow \quad A\,h^2 = 2\,A\,\theta\,h^2 \quad \Rightarrow \quad \theta = \frac{1}{2} \qquad [\because\ A \neq 0,\ h \neq 0]$$

EXERCISE 15 (*b*)

1. Verify Lagrange's mean value theorem for the function
   $f(x) = x^3 - 5\,x^2 - 3\,x$ in $[1, 3]$.
2. Verify Lagrange's mean value theorem for the following functions :
   (*i*) $f(x) = 2\,x^2 - 10\,x + 29$ in $[2, 7]$ (*ii*) $f(x) = x^3 + x^2 - 6\,x$ in $[-1, 4]$
   (*iii*) $f(x) = \log x$ in $[1, e]$ (*iv*) $f(x) = \sin x$ in $\left[0, \frac{\pi}{2}\right]$
   (*v*) $f(x) = \sqrt{x^2 - 4}$ in $[2, 4]$.
3. Verify Lagrange's mean value theorem for the following function :
   $f(x) = (x - 1)\,(x - 2)\,(x - 3)$ in $[0, 4]$
4. If (x) is a quadratic polynomial and a, b are any two numbers, show that $\frac{a + b}{2}$ is the only value of c which satisfies the Mean value theorem in (a, b).
5. Which condition of the hypothesis of the Mean Value Theorem fails to hold for the following functions in the specified interval :
   (*i*) $f(x) = x^{\frac{2}{3}}$ in $[-2, 2]$ (*ii*) $f(x) = \begin{cases} 2\,x + 3 & \text{if } x < 3 \\ 15 - 2\,x & \text{if } x \geq 3 \end{cases}$ in $[-1, 5]$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

6. Use Lagrange's theorem to determine a point P on the curve $y = \sqrt{x^2 - 4}$ defined in the interval [2, 4], where the tangent is parallel to the chord joining the end points on the curve.

7. In Lagrange's Mean Value Theorem, $f(a+h) = f(a) + hf'(a+\theta h)$, $0 < \theta < 1$.

   If $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x$, $a = 0$ and $h = 3$, find values of θ.

8. By using Lagrange's Mean Value Theorem, prove that,

   $|\sin x - \sin y| \le |x - y|$ for all $x, y \in \mathbf{R}$.

ANSWERS

5. (*i*) Not derivable in (– 2, 2) (*ii*) Not derivable in (– 1, 5)

7. $\frac{1}{6}\left(3 \pm \sqrt{3}\right)$

Art-7. Cauchy's Mean Value Theorem

If f and g are two functions such that

(*i*) both are continuous in the closed interval $[a, b]$

(*ii*) both are derivable in the open interval (a, b)

and (*iii*) $g'(x) \neq 0$ for any x in (a, b),

then there exists at least one real number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Proof. First of all, we show that $g(a) \neq g(b)$.

If possible, suppose $g(a) = g(b)$. Now function g is continuous in $[a, b]$, derivable in (a, b) and $g(a) = g(b)$. Therefore g satisfies all the conditions of Rolle's Theorem. Therefore, there exists at least one real number $c \in (a, b)$ such that $g'(c) = 0$, which contradicts the result that $g'(x) \neq 0$ for any x in (a, b).

$\therefore$ our supposition is wrong.

$\therefore$ $g(a) \neq g(b) \Rightarrow \frac{f(b) - f(a)}{g(b) - g(a)}$ is meaningful.

Consider the function $\phi(x) = f(x) + \mathrm{A}\, g(x)$ ...(1)

where A is a constant so chosen that $\phi(a) = \phi(b)$ ...(2)

i.e., $f(a) + \mathrm{A}\, g(a) = f(b) + \mathrm{A}\, g(b)$

i.e., $f(b) - f(a) = -\mathrm{A}\,\{g(b) - g(a)\}$

i.e., $$-\mathrm{A} = \frac{f(b) - f(a)}{g(b) - g(a)} \quad \text{...(3)}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Now the real valued function $\phi(x)$ is such that

(*i*) It is continuous in $[a, b]$ {$\because$ Algebraic sum of continuous functions}

(*ii*) It is derivable in (a, b) {$\because$ Algebraic sum of derivable functions}

(*iii*) $\phi(a) = \phi(b)$ [$\because$ of (2)]

$\therefore$ $\phi(x)$ satisfies all the conditions of Rolle's Theorem in $[a, b]$

$\therefore$ there exists at least one real number $c \in (a, b)$ such that

$$\phi'(c) = 0 \qquad \ldots(4)$$

Differentiating both sides of (1) *w.r.t.* x, we get,

$$\phi'(x) = f'(x) + A\,g'(x) \quad \Rightarrow \quad \phi'(c) = f'(c) + A\,g'(c)$$

$\Rightarrow$ Now $\phi'(c) = 0$ [$\because$ of (4)]

$\Rightarrow$ $f'(c) + A\,g'(c) = 0$

$$\Rightarrow \qquad -A = \frac{f'(c)}{g'(c)} \qquad \ldots(5)$$

From (3) and (5), we get,

$$-\frac{f(b)-f(a)}{g(b)-g(a)} = -\frac{f'(c)}{g'(c)} \quad \Rightarrow \quad \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}, \text{ where } c \in (a, b).$$

Hence the result.

Cor 1. Another form of Cauchy's Mean Value Theorem

If two functions f and g are such that

(*i*) both are continuous in $[a, a+h]$

(*ii*) both are derivable in $(a, a+h)$ and

(*iii*) $g'(x) \neq 0$ for any x in $(a, a+h)$,

then there exists at least one real number θ, $0 < \theta < 1$, such that

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}.$$

Proof. Let $b = a+h$, $c = a+\theta h$

$$\therefore \quad a < c < b \quad \Rightarrow \quad a < a+\theta h < a+h \quad \Rightarrow \quad 0 < \theta h < h \quad \Rightarrow \quad 0 < \theta < 1$$

$$\therefore \quad \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \quad \Rightarrow \quad \frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}.$$

Cor II. Deduction of Lagrange's Mean Value Theorem.

Take $g(x) = x$, $g'(x) = 1$ (exists for each x)

$g(a) = a$, $g(b) = b$, $g'(c) = 1$

$$\therefore \quad \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \quad \Rightarrow \quad \frac{f(b)-f(a)}{b-a} = \frac{f'(c)}{1} \text{ where } c \in (a, b).$$

This result is the same as that of Lagrange's Mean Value Theorem.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Note 1. Cauchy's Mean Value theorem cannot be deduced from Lagrange's Mean Value theorem because θ may be different for $f(x)$ and $g(x)$ in $[a, b]$.

2. Cauchy's Mean value theorem is a slightly more general form of the Lagrange's Mean Value Theorem. Hence Cauchy's Mean Value theorem is also called *Generalized Mean Value theorem* or *the second Mean Value theorem.*

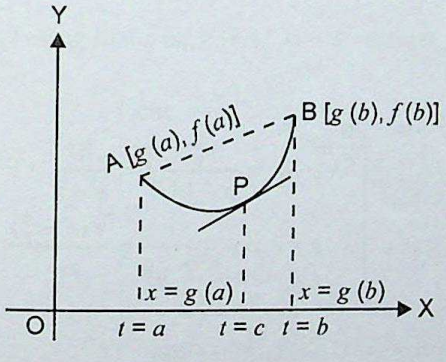
Art-8. Geometrical Significance of Cauchy's Mean Value Theorem

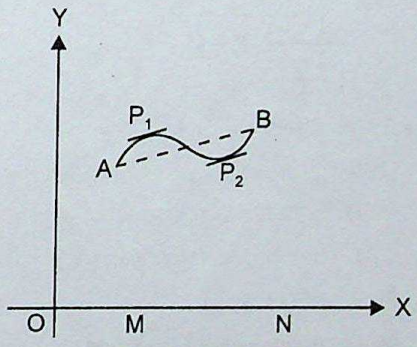
Let a curve be defined parametrically on the closed interval $[a, b]$ by $x = g(t)$ and $y = f(t)$, where $f(t)$ and $g(t)$ are continuous on $[a, b]$ and differentiable in (a, b)

Assuming $g' \neq 0$ in (a, b)

The slope of the tangent at P $= \left(\dfrac{dy}{dx}\right)_{t=c} = \left[\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}\right]_{t=c} = \left(\dfrac{f'(t)}{g'(t)}\right)_{t=c} = \dfrac{f'(c)}{g'(c)}$

The slope of the chord AB $= \dfrac{f(b) - f(a)}{g(b) - g(a)}$





From the figure chord AB is parallel to tangent at P (one such point exists between A and B).

$$\therefore \quad \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \quad \text{where } a < c < b.$$

Hence the result geometrically.

ILLUSTRATIVE EXAMPLES

Example 1. Verify Cauchy's Mean Value Theorem for the functions

$$f(x) = \log x \text{ and } g(x) = \frac{1}{x} \text{ in the interval } [1, e].$$

Sol. $f(x) = \log x$ and $g(x) = \dfrac{1}{x}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$\therefore \quad f'(x) = \dfrac{1}{x}$ and $g'(x) = -\dfrac{1}{x^2}$

Now f and g are two functions such that

(*i*) both are continuous in $[1, e]$

(*ii*) both are derivable in $(1, e)$

(*iii*) $g'(x) \neq 0$ for any x in $(1, e)$

$\therefore$ $f(x)$ and $g(x)$ satisfy all the conditions of Cauchy's Mean Value theorem in $[1, e]$

$\therefore$ by Cauchy's Mean Value theorem

$$\frac{f(e) - f(1)}{g(e) - g(1)} = \frac{f'(c)}{g'(c)} \quad \text{where } 1 < c < e$$

$$\frac{\log e - \log 1}{\frac{1}{e} - \frac{1}{1}} = \frac{\frac{1}{c}}{-\frac{1}{c^2}}$$

$$\frac{1 - 0}{\frac{1 - e}{e}} = -c$$

$$\therefore \quad c = \frac{e}{e - 1} \qquad \left[\begin{array}{l} \because c - 1 = \dfrac{e}{e-1} - 1 = \dfrac{1}{e-1} > 0 \\ e - c = e - \dfrac{e}{e-1} = \dfrac{e(e-2)}{e-1} > 0 \end{array}\right]$$

Clearly $c \in (1, e)$

Hence the theorem is verified.

Example 2. Let $f(x) = \begin{cases} 2, & a \leq x < b \\ 4, & x = b \end{cases}$ and $g(x) = x$, $x \in [a, b]$. Discuss the applicability of Cauchy's Mean Value Theorem to $f(x)$ and $g(x)$ in $[a, b]$.

Sol. $\underset{x \to b-}{\text{Lt}} f(x) = \underset{x \to b-}{\text{Lt}} 2 = 2$

Also $f(b) = 4$

$\therefore \underset{x \to b-}{\text{Lt}} f(x) \neq f(b)$

$\therefore$ $f(x)$ is not continuous at $x = b$

$\therefore$ $f(x)$ is not continuous in $[a, b]$

$\therefore$ Cauchy's Mean Value theorem is not applicable to $f(x)$ and $g(x)$ in $[a, b]$.

Example 3. Use Cauchy's Mean Value Theorem to evaluate $\underset{x \to 1}{\text{Lt}} \dfrac{\cos \frac{\pi x}{2}}{\log \frac{1}{x}}$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Sol. Let $f(x) = \cos \dfrac{\pi x}{2}$, $g(x) = \log x$, $a = x\ (> 0)$, $b = 1$.

Now f and g are two functions of x such that

(*i*) both are continuous in $[x, 1]$

(*ii*) both are derivable in $(x, 1)$

(*iii*) $g'(x) = \dfrac{1}{x} \neq 0$ in $(x, 1)$

$\therefore$ all the conditions of Cauchy's Mean Value Theorem are satisfied.

$\therefore$ by Cauchy's Mean Value Theorem,

$$\frac{f(1) - f(x)}{g(1) - g(x)} = \frac{f'(c)}{g'(c)} \Rightarrow \frac{\cos\frac{\pi}{2} - \cos\frac{\pi x}{2}}{\log 1 - \log x} = \frac{-\frac{\pi}{2}\sin\frac{\pi c}{2}}{\frac{1}{c}}, \text{ where } x < c < 1.$$

$$\Rightarrow \frac{0 - \cos\frac{\pi x}{2}}{\log 1 - \log x} = -\frac{c\pi}{2}\sin\frac{\pi c}{2} \Rightarrow \frac{\cos\frac{\pi x}{2}}{\log\frac{1}{x}} = \frac{c\pi}{2}\sin\frac{\pi c}{2}$$

Taking limits as $x \to 1$, $c \to 1$, we get,

$$\underset{x \to 1}{\text{Lt}} \left(\frac{\cos\frac{\pi x}{2}}{\log\frac{1}{x}} \right) = \frac{\pi}{2}\sin\frac{\pi}{2}$$

$$\therefore \underset{x \to 1}{\text{Lt}} \left(\frac{\cos\frac{\pi x}{2}}{\log\frac{1}{x}} \right) = \frac{\pi}{2} \qquad \left[\because \sin\frac{\pi}{2} = 1\right]$$

EXERCISE 15 (*c*)

1. Verify Cauchy's Mean Value Theorem for the following functions in the indicated intervals. Also find c of this theorem :

(*i*) $f(x) = \cos x$, $g(x) = \sin x$ in $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

(*ii*) $f(x) = \sqrt{x}$, $g(x) = \dfrac{1}{\sqrt{x}}$ in $[a, b]$, $a > 0$

(*iii*) $f(x) = e^x$, $g(x) = e^{-x}$ in $[a, b]$ (J.U. 2016)

(*iv*) $f(x) = x^2$, $g(x) = \sqrt{x}$ in $[1, 4]$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

2. Let $f(x) = x^3$ in $[-1, 1]$

$g(x) = x^4$ in $[-1, 1]$

Is Cauchy's Mean Value theorem applicable to $f(x)$ and $g(x)$ in $[-1, 1]$?

3. Discuss the applicability of Cauchy's mean value theorem for the following functions in the indicated intervals :

(*i*) $f(x) = x^2$, $g(x) = x^3$ in $[-1, 1]$ (*ii*) $f(x) = 2x$, $g(x) = x^4$ in $[-2, 2]$

(*iii*) $f(x) = \begin{cases} 1, & a \le x < b \\ 2, & x = b \end{cases}$; $g(x) = 2x$ in $[a, b]$.

ANSWERS

1. (*i*) 0 (*ii*) $\sqrt{ab}$ (*iii*) $\dfrac{a+b}{2}$ (*iv*) $\left(\dfrac{15}{4}\right)^{\frac{2}{3}}$

2. Not applicable

3. (*i*) Not applicable (*ii*) Not applicable (*iii*) Not applicable.

Art-9. Taylor's Theorem (with Lagrange's form of Remainder)

Statement : If f is a function defined in $[a, b]$ such that

(*i*) $f, f', f'', \ldots f^{(n-1)}(x)$ are continuous in $[a, b]$

(*ii*) $f^{(n)}$ exists in (a, b)

then there exists at least one real number $c \in (a, b)$ such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{\lfloor 2} f''(a) + \ldots + \frac{(b-a)^{n-1}}{\lfloor n-1} f^{(n-1)}(a) + \frac{(b-a)^n}{\lfloor n} f^{(n)}(c)$$

Proof : Consider a function ϕ defined as

$$\phi(x) = f(x) + (b-x)f'(x) + \frac{(b-x)^2}{\lfloor 2} f''(x) + \ldots + \frac{(b-x)^{n-1}}{\lfloor n-1} f^{(n-1)}(x) + \frac{(b-x)^n}{\lfloor n} A \quad \ldots(1)$$

where A is a constant to be determined such that $\phi(a) = \phi(b)$

$$i.e., \; f(a) + (b-a)f'(a) + \frac{(b-a)^2}{\lfloor 2} f''(a) + \ldots + \frac{(b-a)^{n-1}}{\lfloor n-1} f^{(n-1)}(a) + \frac{(b-a)^n}{\lfloor n} A = f(b) \quad \ldots(2)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Now $f(x), f'(x), f''(x), \ldots, f^{(n-1)}(x)$ and $(b-x)^n$, $(n = 1, 2, \ldots, n)$, are continuous in $[a, b]$ and derivable in (a, b).

Therefore, we see that,

(*i*) $\phi(x)$ is continuous in $[a, b]$

(*ii*) $\phi(x)$ is derivable in (a, b) (*iii*) $\phi(a) = \phi(b)$

$\therefore$ $\phi(x)$ satisfies all the conditions of Rolle's Theorem in $[a, b]$ and consequently there exists atleast one real number $c \in (a, b)$ such that

$$\phi'(c) = 0 \qquad \ldots(3)$$

Differentiating both sides of (1) *w.r.t.* x, we get,

$$\phi'(x) = f'(x) + [-f'(x) + (b-x)f''(x)] + \left[-(b-x)f''(x) + \frac{(b-x)^2}{\lfloor 2} f''(x)\right]$$

$$+ \ldots + \left[-\frac{(b-x)^{n-2}}{\lfloor n-2} f^{(n-1)}(x) + \frac{(b-x)^{n-1}}{\lfloor n-1} f^{(n)}(x)\right] + \left[-\frac{(b-x)^{n-1}}{\lfloor n-1} A\right]$$

$$\therefore \quad \phi'(x) = \frac{(b-x)^{n-1}}{\lfloor n-1} f^{(n)}(x) - \frac{(b-x)^{n-1}}{\lfloor n-1} A$$

$$\Rightarrow \quad \phi'(x) = \frac{(b-x)^{n-1}}{\lfloor n-1} [f^{(n)}(x) - A] \quad \Rightarrow \quad \phi'(c) = \frac{(b-c)^{n-1}}{\lfloor n-1} [f^{(n)}(c) - A]$$

$$\Rightarrow \quad 0 = \frac{(b-c)^{n-1}}{\lfloor n-1} [f^{(n)}(c) - A] \qquad [\because \text{ of } (3)]$$

$$\Rightarrow \quad 0 = f^{(n)}(c) - A \qquad [\because \ b - c \neq 0 \text{ as } a < c < b]$$

$$\Rightarrow \quad A = f^{(n)}(c)$$

Substituting this value of A in (2), we get,

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{\lfloor 2} f''(a) + \ldots + \frac{(b-a)^{n-1}}{\lfloor n-1} f^{(n-1)}(a)$$

$$+ \frac{(b-a)^n}{\lfloor n} f^{(n)}(c)$$

Hence the result.

Note 1. $\frac{(b-a)^n}{\lfloor n} f^{(n)}(c)$ is called **Lagrange's Remainder** after n terms and is generally denoted by R_n.

Note 2. If $n = 1$, this theorem becomes Lagrange's Mean Value Theorem.

Note 3. This theorem is also called Higher Mean Value Theorem.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Note 4. Another form of above theorem

If f is a function defined in $[a, a+h]$ such that

(*i*) $f, f', f'', \ldots, f^{(n-1)}$ are continuous in $[a, a+h]$

(*ii*) $f^{(n)}$ exists in $(a, a+h)$,

then there exists at least one real number θ, $0 < \theta < 1$, such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2} f''(a) + \ldots + \frac{h^{n-1}}{\lfloor n-1} f^{(n-1)}(a) + \frac{h^n}{\lfloor n} f^{(n)}(a+\theta h)$$

[Put $b = a+h$, $c = a+\theta h$]

Here $R_n = \dfrac{h^n}{\lfloor n} f^{(n)}(a+\theta h)$, $0 < \theta < 1$, is called Lagrange's Remainder after n terms.

Note 5. Maclaurin's Theorem with Lagrange's Form of Remainder.

Put $a = 0$, $h = x$ in the 2nd Form.

$$\therefore\ f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \ldots$$

$$+ \frac{x^{n-1}}{\lfloor n-1} f^{(n-1)}(0) + \frac{x^n}{\lfloor n} f^{(n)}(\theta x) \text{ where } 0 < \theta < 1$$

This is called Maclaurin's Theorem with Lagrange's Form of Remainder after n terms. $R_n = \dfrac{x^n}{\lfloor n} f^{(n)}(\theta x)$ is called Lagrange's Remainder after n terms in the Maclaurin's expansion of $f(x)$.

Art-10. Taylor's Theorem (with Cauchy's Form of Remainder)

Statement : If f is a function defined in $[a, b]$ such that

(*i*) $f, f', f'', \ldots, f^{(n-1)}$ are continuous in $[a, b]$

(*ii*) f^n exists in (a, b),

then there exists atleast one real number θ, $0 < \theta < 1$, such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{\lfloor 2} f''(a) + \ldots + \frac{(b-a)^{n-1}}{\lfloor n-1} f^{(n-1)}(a)$$

$$+ \frac{(b-a)^n}{\lfloor n-1} (1-\theta)^{n-1} f^{(n)}[a+\theta(b-a)]$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Proof. Consider a function ϕ defined as

$$\phi(x) = f(x) + (b-x)f'(x) + \frac{(b-x)^2}{\lfloor 2} f''(x) + \ldots$$

$$+ \frac{(b-x)^{n-1}}{\lfloor n-1} f^{(n-1)}(x) + (b-x)\,\mathrm{A} \qquad \ldots(1)$$

where A is a constant to be determined such that $\phi(a) = \phi(b)$

$$i.e.,\ f(a) + (b-a)f'(a) + \frac{(b-a)^2}{\lfloor 2} f''(a) + \ldots$$

$$+ \frac{(b-a)^{n-1}}{\lfloor n-1} f^{(n-1)}(a) + (b-a)\,\mathrm{A} = f(b) \qquad \ldots(2)$$

Now $f(x)$, $f'(x)$, $f''(x)$, ..., $f^{(n-1)}(x)$ and $(b-x)^n$, $(n = 1, 2, \ldots, n-1)$, are continuous in $[a, b]$ and derivable in (a, b).

Therefore, we see that,

(*i*) $\phi(x)$ is continuous in $[a, b]$

(*ii*) $\phi(x)$ is derivable in (a, b)

(*iii*) $\phi(a) = \phi(b)$

$\therefore$ $\phi(x)$ satisfies all the conditions of Rolle's Theorem in $[a, b]$ and consequently there exists at least one real number $\theta \in (0, 1)$ such that

$$\phi'[a + \theta(b-a)] = 0 \qquad \ldots(3)$$

Differentiating both sides of (1) *w.r.t.* x, we get,

$$\phi'(x) = f'(x) + [-f'(x) + (b-x)f''(x)] + \left[-(b-x)f''(x) + \frac{(b-x)^2}{\lfloor 2} f'''(x)\right] + \ldots$$

$$+ \left[-\frac{(b-x)^{n-2}}{\lfloor n-2} f^{(n-1)}(x) + \frac{(b-x)^{n-1}}{\lfloor n-1} f^{(n)}(x)\right] - \mathrm{A}$$

$$\text{or}\quad \phi'(x) = \frac{(b-x)^{n-1}}{\lfloor n-1} f^{(n)}(x) - \mathrm{A}$$

$$\Rightarrow\ \phi'[a + (b-a)\theta] = \frac{[b-a-(b-a)\theta]^{n-1}}{\lfloor n-1} f^{(n)}[a + (b-a)\theta] - \mathrm{A}$$

$$\Rightarrow\ 0 = \frac{(b-a)^{n-1}(1-\theta)^{n-1}}{\lfloor n-1} f^{(n)}[a + (b-a)\theta] - \mathrm{A} \qquad [\because \text{ of } (3)]$$

$$\Rightarrow\ \mathrm{A} = \frac{(b-a)^{n-1}(1-\theta)^{n-1}}{\lfloor n-1} f^{(n)}[a + (b-a)\theta]$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Substituting the value of A in (2), we get,

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{\lfloor 2} f''(a) + \ldots$$

$$+ \frac{(b-a)^{n-1}}{\lfloor n-1} f^{(n-1)}(a) + \frac{(b-a)^n}{\lfloor n-1} (1-\theta)^{n-1} f^{(n)}[a + \theta(b-a)]$$

Hence the result.

Note 1. $\frac{(b-a)^n}{\lfloor n-1} (1-\theta)^{n-1} f^{(n)}[a+\theta(b-a)]$ is called **Cauchy's Remainder** after n terms and is generally denoted by R_n.

Note 2. Another form of above theorem

If f is a function defined in $[a, a+h]$ such that

(*i*) $f, f', f'', \ldots, f^{(n-1)}$ are continuous in $[a, a+h]$

(*ii*) $f^{(n)}$ exists in $(a, a+h)$

then there exists at least one real number θ, $0 < \theta < 1$, such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2} f''(a) + \ldots + \frac{h^{n-1}}{\lfloor n-1} f^{(n-1)}(a)$$

$$+ \frac{h^n}{\lfloor n-1} (1-\theta)^{n-1} f^{(n)}(a + \theta h)$$

[Put $b = a+h$, $c = a + \theta h$]

Here $R_n = \frac{h^n (1-\theta)^{n-1}}{\lfloor n-1} f^{(n)}(a + \theta h)$ is called **Cauchy's Remainder** after n terms.

Note 3. Maclaurin's Theorem with Cauchy's Form of Remainder.

Put $a = 0$, $h = x$ in the 2nd Form.

$$\therefore\ f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots + \frac{x^{n-1}}{\lfloor n-1} f^{(n-1)}(0)$$

$$+ \frac{x^n}{\lfloor n-1} (1-\theta)^{n-1} f^{(n)}(\theta x)$$

where $0 < \theta < 1$

This is called Maclaurin's Theorem with Cauchy's Form of remainder after n terms.

$R_n = \frac{x^n}{\lfloor n-1} (1-\theta)^{n-1} f^{(n)}(\theta x)$, $0 < \theta < 1$ is called Cauchy's Remainder after n terms in the Maclaurin's expansion of $f(x)$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

ILLUSTRATIVE EXAMPLES

Example 1. Apply Taylor's Theorem with Lagrange's form of remainder to the function :

$$f(x) = \sin x \text{ in } \left[\frac{\pi}{2}, x\right].$$

Sol. By Taylor's Theorem with Lagrange's form of remainder to the function $f(x)$ in $\left[\frac{\pi}{2}, x\right]$, we have

$$f(x) = f\left(\frac{\pi}{2}\right) + \left(x - \frac{\pi}{2}\right) f'\left(\frac{\pi}{2}\right) + \frac{\left(x - \frac{\pi}{2}\right)^2}{\lfloor 2} f''\left(\frac{\pi}{2}\right)$$

$$+ \frac{\left(x - \frac{\pi}{3}\right)^3}{\lfloor 3} f'''\left(\frac{\pi}{2}\right) + \ldots\ldots + \frac{\left(x - \frac{\pi}{2}\right)^{n-1}}{\lfloor n-1} f^{n-1}\left(\frac{\pi}{2}\right) + \frac{\left(x - \frac{\pi}{2}\right)^n}{\lfloor n} f^n\left(\frac{\pi}{2} + \theta\left(x - \frac{\pi}{2}\right)\right)$$

when $0 < \theta < 1$ ...(1)

$f(x) = \sin x$

$f'(x) = \cos x = \sin\left(x + \frac{\pi}{2}\right)$

$f''(x) = \cos\left(x + \frac{\pi}{2}\right) = \sin\left(x + 2.\frac{\pi}{2}\right)$

.....

$f^{n-1}(x) = \sin\left(x + (n-1)\frac{\pi}{2}\right)$

$f^n(x) = \sin\left(x + n\frac{\pi}{2}\right)$

$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$

$f'\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \sin\pi = 0$

$f''\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2} + 2\frac{\pi}{2}\right) = \sin\frac{3\pi}{2} = -1$

.....

$f^{n-1}\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2} + (n-1)\frac{\pi}{2}\right) = \sin n\frac{\pi}{2}$

$f^n\left[\frac{\pi}{2} + \theta\left(x - \frac{\pi}{2}\right)\right]$

$= \sin\left[(n+1)\frac{\pi}{2} + \theta\left(x - \frac{\pi}{2}\right)\right]$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bhatinda Campus

$\therefore$ from (1), we have

$$\sin x = 1 + \left(x - \frac{\pi}{2}\right).0 + \frac{\left(x - \frac{\pi}{2}\right)^2}{\lfloor 2}(-1) + \ldots$$

$$+ \frac{\left(x - \frac{\pi}{2}\right)^{n-1}}{\lfloor n-1}.\left(\sin n\frac{\pi}{2}\right) + \frac{\left(x - \frac{\pi}{2}\right)^{n}}{\lfloor n}\sin\left[(n+1)\frac{\pi}{2} + \theta\left(x - \frac{\pi}{2}\right)\right]$$

or $$\sin x = 1 - \frac{\left(x - \frac{\pi}{2}\right)^2}{\lfloor 2} + \ldots + \frac{\left(x - \frac{\pi}{2}\right)^{n-1}}{\lfloor n-1}\sin\frac{n\pi}{2}$$

$$+ \frac{\left(x - \frac{\pi}{2}\right)^{n}}{\lfloor n}\sin\left[(n+1)\frac{\pi}{2} + \theta\left(x - \frac{\pi}{2}\right)\right]$$

Example 2. If $f(x) = x^3 - 2x + 5$, find the value of $f(2{\cdot}001)$ with the help of Taylor's Theorem. Find an approximate change in the value of $f(x)$ when x changes from 2 to 2·001.

Sol. By Taylor's Theorem

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2}f''(x) + \frac{h^3}{\lfloor 3}f'''(x) + \ldots$$

Put $x = 2,\ h = 0{\cdot}001$

$$\therefore\ f(2{\cdot}001) = f(2) + 0{\cdot}001 \times f'(2) + \frac{(0{\cdot}001)^2}{\lfloor 2}f''(2) + \frac{(0{\cdot}001)^3}{\lfloor 3} \times f'''(2) + \ldots \quad \ldots(1)$$

Now $f(x) = x^3 - 2x + 5 \quad \Rightarrow \quad f(2) = 8 - 4 + 5 = 9$

$\therefore\ f'(x) = 3x^2 - 2 \quad \Rightarrow \quad f'(2) = 12 - 2 = 10$

$f''(x) = 6x \quad \Rightarrow \quad f''(2) = 12$

$f'''(x) = 6 \quad \Rightarrow \quad f'''(2) = 6$

$f^{iv}(x) = 0 \quad \Rightarrow \quad f^{iv}(2) = 0$

From (1), we get,

$$f(2{\cdot}001) = 9 + 0{\cdot}001 \times 10 + \frac{0{\cdot}000001}{2} \times 12 + \frac{0{\cdot}000000001}{6} \times 6$$

$$= 9 + 0{\cdot}01 + 0{\cdot}000006 + 0{\cdot}000000001$$

$$= 9{\cdot}010060001 = 9{\cdot}01 \text{ approximately.}$$

$\therefore$ required change $= f(2{\cdot}001) - f(2) = 9{\cdot}01 - 9 = 0{\cdot}01$ approximately.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

 Gandhi Memorial College Of Education Bantalab Jammu

Example 3. Calculate the approximate value of $\sqrt{26}$ to three decimal places by Taylor's Expansion.

Sol. Let $f(x+h) = \sqrt{x+h}$

Putting $h = 0$, we get, $f(x) = \sqrt{x}$

$$\therefore\ f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}},\ f''(x) = -\frac{1}{4}x^{-\frac{3}{2}},\ f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}, \ldots$$

By Taylor's Expansions

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2}f''(x) + \frac{h^3}{\lfloor 3}f'''(x) + \ldots$$

$$\therefore\ \sqrt{x+h} = \sqrt{x} + \frac{h}{2\sqrt{x}} - \frac{h^2}{8x\sqrt{x}} + \frac{h^3}{16x^2\sqrt{x}} + \ldots$$

Put $x = 25,\ h = 1$

$$\therefore\ \sqrt{26} = 5 + \frac{1}{2\times 5} - \frac{1}{8\times 25\times 5} + \frac{1}{16\times 625\times 5} + \ldots$$

$$= 5 + \frac{1}{10} - \frac{1}{1000} + \frac{1}{50000} + \ldots$$

$$= 5 + 0{\cdot}1 - 0{\cdot}001 + 0{\cdot}00002 + \ldots.$$

$$= 5{\cdot}09902 \text{ nearly} = 5{\cdot}099 \text{ nearly.}$$

Example 4. Expand $\sin x$ in ascending powers of $\left(x - \frac{\pi}{2}\right)$. (J.U. 2016)

Sol. Let $f(x) = \sin x$

Now $f(x) = f\left[\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right]$

$\therefore\ f(x) = f(a+h)$ where $a = \frac{\pi}{2},\ h = x - \frac{\pi}{2}$

$$\Rightarrow\ f(x) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2}f''(a) + \frac{h^3}{\lfloor 3}f'''(a) + \frac{h^4}{\lfloor 4}f^{iv}(a) + \ldots$$

$$\Rightarrow\ \sin x = f\left(\frac{\pi}{2}\right) + \left(x - \frac{\pi}{2}\right)f'\left(\frac{\pi}{2}\right) + \frac{\left(x - \frac{\pi}{2}\right)^2}{\lfloor 2}f''\left(\frac{\pi}{2}\right)$$

$$+ \frac{\left(x - \frac{\pi}{2}\right)^3}{\lfloor 3}f'''\left(\frac{\pi}{2}\right) + \frac{\left(x - \frac{\pi}{2}\right)^4}{\lfloor 4}f^{iv}\left(\frac{\pi}{2}\right) + \ldots \qquad \ldots(1)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Now $f(x) = \sin x \quad \Rightarrow \quad f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$

$f'(x) = \cos x \quad \Rightarrow \quad f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$

$f''(x) = -\sin x \quad \Rightarrow \quad f''\left(\frac{\pi}{2}\right) = -1$

$f'''(x) = -\cos x \quad \Rightarrow f'''\left(\frac{\pi}{2}\right) = 0$

$f^{iv}(x) = \sin x \quad \Rightarrow \quad f^{iv}\left(\frac{\pi}{2}\right) = 1$

and so on.

$\therefore$ from (1), we get,

$$\sin x = 1 - \frac{\left(x-\frac{\pi}{2}\right)^2}{\lfloor 2} + \frac{\left(x-\frac{\pi}{2}\right)^4}{\lfloor 4} - \ldots$$

Example 5. Use Maclaurin's Theorem with Lagrange's form of remainder to expand $\sin x$ as far as the nth term in terms of ascending powers of x.

Sol. By Maclaurin's Theorem with Lagrange's form of remainder to the function $f(x) = \sin x$, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \ldots + \frac{x^{n-1}}{\lfloor n-1} f^{(n-1)}(0)$$

$$+ \frac{x^n}{\lfloor n} f^{(n)}(\theta x), \quad 0 < \theta < 1 \qquad \ldots(1)$$

$f(x) = \sin x$	$f(0) = \sin 0 = 0$
$f'(x) = \cos x = \sin\left(x + \frac{\pi}{2}\right)$	$f'(0) = \sin\frac{\pi}{2} = 1$
$f''(x) = \cos\left(x + \frac{\pi}{2}\right) = \sin\left(x + 2.\frac{\pi}{2}\right)$	$f''(0) = \sin\pi = 0$
$f'''(x) = \cos\left(x + 2.\frac{\pi}{2}\right) = \sin\left(x + 3.\frac{\pi}{2}\right)$	$f'''(0) = \sin\frac{3\pi}{2} = -1$
... ...	... ...
$f^{(n-1)}(x) = \sin\left[x + \frac{(n-1)\pi}{2}\right]$	$f^{(n-1)}(0) = \sin\frac{(n-1)\pi}{2}$
$f^{(n)}(x) = \sin\left[x + \frac{n\pi}{2}\right]$	$f^{(n)}(\theta x) = \sin\left[\theta x + \frac{n\pi}{2}\right]$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

 Gandhi Memorial College Of Education Bantalab Jammu

$\therefore$ from (1), we get,

$$\sin x = 0 + x(1) + \frac{x^2}{\lfloor 2}(0) + \frac{x^3}{\lfloor 3}(-1) + \ldots + \frac{x^{n-1}}{\lfloor n-1} \sin\frac{(n-1)\pi}{2}$$

$$+ \frac{x^n}{\lfloor n} \sin\left(\theta x + \frac{n\pi}{2}\right)$$

or $\quad \sin x = x - \dfrac{x^3}{\lfloor 3} + \dfrac{x^5}{\lfloor 5} - \ldots + \dfrac{x^{n-1}}{\lfloor n-1} \sin\dfrac{(n-1)\pi}{2} + \dfrac{x^n}{\lfloor n} \sin\left(\theta x + \dfrac{n\pi}{2}\right), \; 0 < \theta < 1.$

Example 6. If $f''(x)$ is a continuous function of x, then prove that

$$\underset{h \to 0}{\text{Lt}} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} = f''(x).$$

Sol. Since $f''(x)$ exists

$\therefore$ $f'(x)$ is derivable

$\Rightarrow$ $f'(x)$ is continuous function of x

[$\because$ a function derivable at a point is always continuous at that point]

Applying Taylor's Theorem with Lagrange's form of remainder after two terms to $f(x)$ in $[x, x+2h]$ we have,

$$f(x+2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{\lfloor 2} f''(x + 2\theta_1 h) \qquad \ldots(1)$$

where $0 < \theta_1 < 1$.

Again applying Taylor's Theorem with Lagrange's form of remainder after two terms to $f(x)$ in $[x, x+h]$, we have,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2} f''(x + \theta_2 h) \qquad \ldots(2)$$

where $0 < \theta_2 < 1$.

Now $\quad \underset{h \to 0}{\text{Lt}} \dfrac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$

$$= \underset{h \to 0}{\text{Lt}} \frac{1}{h^2}\left[f(x) + 2hf'(x) + \frac{4h^2}{2} f''(x + 2\theta_1 h) - 2f(x)\right.$$

$$\left. - 2hf'(x) - \frac{2h^2}{\lfloor 2} f''(x + \theta_2 h) + f(x)\right]$$

$$= \underset{h \to 0}{\text{Lt}} \frac{1}{h^2}\left[2h^2 f''(x + 2\theta_1 h) - h^2 f''(x + \theta_2 h)\right]$$

$$= \underset{h \to 0}{\text{Lt}} \left[2f''(x + 2\theta_1 h) - f''(x + \theta_2 h)\right] = 2f''(x) - f''(x) = f''(x)$$

$$\therefore \quad \underset{h \to 0}{\text{Lt}} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} = f''(x).$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantala, Kolkata

Example 7. Show that the number θ which occurs in the Taylor's theorem with Lagrange's form of remainder after n terms approaches the limit $\frac{1}{n+1}$ as h approaches zero provided $f^{(n+1)}(x)$ is continuous and different from zero at $x = a$.

Sol. Since $f^{(n+1)}(x)$ is continuous at $x = a$

$\therefore$ function f is defined in $[a, a+h]$ such that

(*i*) $f, f', f'', \ldots, f^{(n)}$ are continuous in $[a, a+h]$

(*ii*) $f^{(n+1)}$ exists in $(a, a+h)$

$\therefore$ applying Taylor's Theorem with Lagrange's form of remainder after n terms and $(n+1)$ terms to the function f, we have,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2} f''(a) + \ldots + \frac{h^{n-1}}{\lfloor n-1} f^{(n+1)}(a) + \frac{h^n}{\lfloor n} f^{(n)}(a+\theta h) \quad \ldots(1)$$

where $0 < \theta < 1$ and $\theta \in \mathbf{R}$

and $$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2} f''(a) + \ldots + \frac{h^n}{\lfloor n} f^{(n)}(a) + \frac{h^{n+1}}{\lfloor n+1} f^{(n+1)}(a+\theta_1 h) \quad \ldots(2)$$

where $0 < \theta_1 < 1$ and $\theta_1 \in \mathbf{R}$.

Equating values of $f(a+h)$ from (1) and (2), we get,

$$\frac{h^n}{\lfloor n} f^{(n)}(a+\theta h) = \frac{h^n}{\lfloor n} f^{(n)}(a) + \frac{h^{n+1}}{\lfloor n+1} f^{(n+1)}(a+\theta_1 h)$$

or $$f^{(n)}(a+\theta h) = f^{(n)}(a) + \frac{h}{n+1} f^{(n+1)}(a+\theta_1 h) \quad \ldots(3)$$

Again applying Lagrange's Mean Value Theorem to the function $f^{(n)}(x)$ in $[a, a+\theta h]$, we have

$$f^{(n)}(a+\theta h) = f^{(n)}(a) + \theta h f^{(n+1)}(a+\theta_2 \theta h) \quad \ldots(4)$$

where $0 < \theta_2 < 1$, $\theta_2 \in \mathbf{R}$.

From (3) and (4), we get,

$$\theta h f^{(n+1)}(a+\theta_2 \theta h) = \frac{h}{n+1} f^{(n+1)}(a+\theta_1 h)$$

$$\Rightarrow \quad \theta = \frac{1}{n+1} \frac{f^{(n+1)}(a+\theta_1 h)}{f^{(n+1)}(a+\theta_2 \theta h)}$$

$$\Rightarrow \quad \underset{h \to 0}{\text{Lt}}\ \theta = \frac{1}{n+1} \underset{h \to 0}{\text{Lt}} \frac{f^{(n+1)}(a+\theta_1 h)}{f^{(n+1)}(a+\theta_2 \theta h)}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

$$\Rightarrow \quad \underset{h \to 0}{\text{Lt}} \ \theta = \frac{1}{n+1} \frac{f^{(n+1)}(a)}{f^{(n+1)}(a)}$$

$$\Rightarrow \quad \underset{h \to 0}{\text{Lt}} \ \theta = \frac{1}{n+1} \qquad [\because f^{(n+1)}(a) \neq 0]$$

which is the required result.

EXERCISE 15 (*d*)

1. Apply Taylor's Theorem with Lagrange's form of remainder to the function $f(x) = \cos x$ in $\left[\frac{\pi}{2}, x\right]$.

2. Apply Taylor's Theorem to obtain the following functions in the indicated form :

   (*i*) $e^{\cos x} = 1 - \left(x - \frac{\pi}{2}\right) + \frac{\left(x - \frac{\pi}{2}\right)^2}{2} - \frac{\left(x - \frac{\pi}{2}\right)^4}{8} + \ldots, a = \frac{\pi}{2}$

   (*ii*) $\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots, a = 1$

   (*iii*) $f(x) = -9 - 26(x-2) - 13(x-2)^2 + (x-2)^3 + (x-2)^4$
   where $f(x) = x^4 - 7x^3 + 5x^2 + 6x - 1$

   (*iv*) $e^{ax} \cos bx = 1 + ax + \frac{a^2 - b^2}{\lfloor 2} x^2 + \frac{a(a^2 - 3b^2)}{\lfloor 3} x^3 + \ldots$

3. If $f(x) = x^3 - 6x^2 + 7$; find the value of $f\left(\frac{21}{20}\right)$ by Taylor's Theorem.

4. If $f(x) = 5x^3 - 3x^2 + 7x - 8$, find the value of $f(3{\cdot}001)$ with the help of Taylor's Theorem. Find an approximate change in the value of $f(x)$ when x changes from 3 to 3·001.

5. Calculate the approximate value of $\sqrt{10}$ to four decimal places by taking the first four terms of an appropriate Taylor's expansion.

6. Calculate the approximate value of $\sqrt{17}$ to four decimal places by taking first three terms of a Taylor's Expansion.

7. Expand $\sin x$ in powers of $(x - a)$.

8. If $0 < x \leq 2$, then prove that

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

9. Use Taylor's Theorem to express the polynomial $2x^3 + 7x^2 + x - 6$ in powers of $(x - 2)$.

10. Use Maclaurin's Theorem to show that

   (*i*) $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \ldots + \frac{x^{n-1}}{\lfloor n-1} \cos\frac{(n-1)\pi}{2}$

$$+ \frac{x^n}{\lfloor n} \cos\left(\frac{n\pi}{2} + \theta x\right), \ 0 < \theta < 1$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

(*ii*) $\sin ax = ax - \frac{a^3x^3}{\lfloor 3} + \frac{a^5x^5}{\lfloor 5} - \ldots + \frac{a^{n-1}x^{n-1}}{\lfloor n-1} \sin\frac{(n-1)\pi}{2}$

$$+ \frac{a^n x^n}{\lfloor n} \sin\left(\frac{n\pi}{2} + a\theta x\right), \ 0 < \theta < 1$$

(*iii*) $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots + \frac{x^{n-1}}{\lfloor n-1} + \frac{x^n}{\lfloor n} e^{\theta x}, \ 0 < \theta < 1$

(*iv*) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^{n-1}}{n} - \frac{x^n}{(1+\theta x)^n}, \ 0 < \theta < 1.$

(J.U. 2016)

11. Obtain the first three non-zero terms in the Maclaurin's formula for the functions :

(*i*) $\tan x$ (*ii*) $\sec x$ (*iii*) $\sin^{-1} x$

(*iv*) $\tan^{-1} x$ (*v*) $\sin x^2$ (*vi*) $e^{\sin x}$.

12. If $f(x+h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2} f''(x + \theta h)$, find the value of θ as $x \to a$ where $f(x) = (x-a)^{\frac{5}{2}}$.

ANSWERS

3. 1·542625 **4.** 121·124 nearly ; 0·124

5. 3·1623 nearly **6.** 4·1231 nearly

7. $\sin a + (x-a)\cos a - \frac{(x-a)^2}{\lfloor 2}\sin a - \frac{(x-a)^3}{\lfloor 3}\cos a + \ldots$

9. $40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$

11. (*i*) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots.$ (*ii*) $\sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \ldots\ldots$

(*iii*) $\sin^{-1} x = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \ldots\ldots$

(*iv*) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots$ (*v*) $\sin x^2 = x^2 - \frac{x^6}{\lfloor 3} + \frac{x^{10}}{\lfloor 5} - \ldots\ldots$

(*vi*) $e^{\sin x} = 1 + x + \frac{x^2}{\lfloor 2} + \ldots\ldots$

12. $\frac{64}{225}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

16

ENVELOPES

Art-1. One Parameter Family of Curves

If $f(x, y, \alpha)$ be any function of three variables, then the equation $f(x, y, \alpha) = 0$, determines a curves corresponding to each particular value of α.

The totality of these curves, obtained by giving different values to α, is said to be a **one parameter family of curves.**

The variable α, which is different for different curves, is said to be the **parameter** for the family.

Examples

(1) The equation $x^2 + y^2 - 2\alpha x = 0$ determines a family of circles with their centres on x-axis and passing through the origin. Here α is the parameter.

(2) The equation $y = mx - 2am - am^3$ determines a family of straight lines which are normals to the parabola $y^2 = 4ax$. Here m is the parameter.

Art-2. Envelope

Before defining envelope of a one parameter family of curves, we introduce the concept by means of an example.

Consider the family of straight lines

$$y = mx + \frac{a}{m} \qquad \ldots(1)$$

where m is the parameter and a is some constant.

The two members of this family corresponding to the parametric values m_1 and δm_1 of the parameter m are

$$y = m_1 x + \frac{a}{m_1} \qquad \ldots(2)$$

$$y = (m_1 + \delta m_1)x + \frac{a}{m_1 + \delta m_1} \qquad \ldots(3)$$

Subtracting (2) from (3), we get,

$$0 = (m_1 + \delta m_1 - m_1)x + a\left(\frac{1}{m_1 + \delta m_1} - \frac{1}{\delta m_1}\right)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantala Jaipur

or $\quad 0 = x.\,\delta m_1 - \dfrac{a.\delta m_1}{m_1(m_1+\delta m_1)}$ $\quad$ or $\quad 0 = x - \dfrac{a}{m_1(m_1+\delta m_1)}$

$$\therefore \quad x = \frac{a}{m_1(m_1+\delta m_1)}$$

Putting this value of x in (2), we get,

$$y = \frac{a}{m_1+\delta m_1} + \frac{a}{m_1} = a\left(\frac{1}{m_1+\delta m_1} + \frac{1}{m_1}\right) = \frac{a(2m_1+\delta m_1)}{m_1(m_1+\delta m_1)}$$

$\therefore$ two lines (2)and (3) interest at (x, y), where

$$x = \frac{a}{m_1(m_1+\delta m_1)},\ y = \frac{a(2m_1+\delta m_1)}{m_1(m_1+\delta m_1)}$$

$\therefore$ two lines (2)and (3) intersect at (x, y), where

$$x = \frac{a}{m_1(m_1+\delta m_1)},\ y = \frac{a(2m_1+\delta m_1)}{m_1(m_1+\delta m_1)}$$

As $\delta m_1 \to 0$ *i.e.,* as line (3) tends to coincide with the line (2), this point of intersection goes on changing its position on the line (2) and, in the limit, tends to the point $\left(\dfrac{a}{m_1^2}, \dfrac{2a}{m_1^2}\right)$, which lies on (2).

This point is the limiting position of the point of the intersection of the line (2) with another of the family when the latter tends to coincide with former.

There will lie a point, similarly obtained, one every line of the family. The locus of such points is called the envelope of the given family of lines. For finding the locus for the family of lines (2), we eliminate m from the equations $x = \dfrac{a}{m^2}, y = \dfrac{2a}{m}$ and obtain $y^2 = 4ax$ as the envelope of the given family of lines.

Definition of Envelope. The envelope of a one-parameter family of curves is the locus of the limiting position of the points of intersection of any two curves of the family when one of them tends to coincide with the other which is kept fixed.

Art-3. Determination of Envelope

Obtain the envelope of a one-parameter family of curves.

Proof. Let the equation of a one parameter family of curves be

$$f(x, y, \alpha) = 0 \qquad \ldots(1)$$

Consider any two members

$$f(x, y, \alpha) = 0 \qquad \ldots(2)$$

and $\quad f(x, y, \alpha + \delta\alpha) = 0 \qquad \ldots(3)$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

The points common to the two curves (1) and (2) satisfy the equation

$$f(x, y, \alpha + \delta\alpha) - f(x, y, \alpha) = 0$$

$$\therefore \quad \frac{f(x, y, \alpha + \delta\alpha) - f(x, y, \alpha)}{\delta\alpha} = 0 \qquad \ldots(4)$$

Let $\delta\alpha \to 0$. Therefore the limiting positions of the points common to (2) and (3) satisfy the equation

$$f_\alpha(x, y, \alpha) = 0 \qquad \ldots(5)$$

which is the limit of (4).

Thus the co-ordinates of the points on the envelope satisfy the equations (1) and (5).

Let the elimination of α between (1) and (5) give the equation

$$\phi(x, y) = 0$$

This is the required envelope.

Note. For obtaining the envelope of the family of curves $f(x, y, \alpha) = 0$, we eliminate α between $f(x, y, \alpha) = 0$ and $f_\alpha(x, y, \alpha) = 0$ where $f_\alpha(x, y, \alpha)$ is the partial derivative of $f(x, y, \alpha)$ w.r.t. α.

If, on solving equations (1) and (5) for x, y, we obtain

$$x = \phi(\alpha) \qquad \ldots(6)$$

$$y = \psi(\alpha) \qquad \ldots(7)$$

Then (6) and (7) are the parametric equations of the envelope where α is the parametrer.

Art-4. If the equation to a one-parameter family of curves is a quadratic in the parameter, then show that the envelope is obtained by equating its discriminant to zero.

Proof. Let the equation to the family of curves be

$$A\alpha^2 + B\alpha + C = 0 \qquad \ldots(1)$$

where α is the parameter and A, B, C are functions of x and y.

Differentiating (1) partially w.r.t. α, we have

$$2A\alpha + B = 0 \quad \text{or} \quad \alpha = -\frac{B}{2A} \qquad \ldots(2)$$

Eliminating α between (1) and (2), we get,

$$A\left(-\frac{B}{2A}\right)^2 + B\left(-\frac{B}{2A}\right) + C = 0$$

$$\text{or} \quad \frac{B^2}{4A} - \frac{B^2}{2A} + C = 0 \quad \text{or} \quad B^2 - 2B^2 + 4AC = 0$$

$$\text{or} \quad B^2 - 4AC = 0$$

which gives the equation of envelope.

Hence the result.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College of Education Bantalab Jammu

Art-5. Show that the locus of the singular points of the curves of a given family is a part of its envelope.

Proof. Let the family of curves be

$$f(x, y, \alpha) = 0 \qquad \ldots(1)$$

where α is the parameter.

Differentiating (10 partially w.r.t. t, we get,

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial \alpha} d\alpha = 0 \qquad \ldots(2)$$

At a singular point of (1), we have

$$\frac{\partial f}{\partial x} = 0, \ \frac{\partial f}{\partial y} = 0$$

$\therefore$ from (2), $\dfrac{\partial f}{\partial \alpha} = 0$ $\qquad \ldots(3)$

$\therefore$ singular points of (1) satisfies both the equations (1) and (3), and hence they lie on envelope of family (1).

Hence the result.

Art-6. Prove that, in general, the envelope of a family of curves touches each member of the family.

Proof. Let the equation of family of curves be

$$f(x, y, \alpha) = 0 \qquad \ldots(1)$$

Its envelope is obtained by eliminating α between

$$f(x, y, \alpha) = 0 \qquad \ldots(2)$$

and $\qquad f_\alpha(x, y, \alpha) = 0 \qquad \ldots(3)$

Suppose, by solving (2) and (3) for x and y in terms of α, we get

$$\left.\begin{aligned} x &= \phi(\alpha) \\ y &= \psi(\alpha) \end{aligned}\right\} \qquad \ldots(4)$$

These are, then the parametric equations of the envelope, the parameter being α.

Now (1) is satisfied by relations (4) for all values of α. Hence, differentiating (1) *w.r.t* α, regarding L.H.S. of (1) as a composite function of α, by virtue of relations (4), we get,

$$\frac{\partial f}{\partial x}\frac{dx}{d\alpha} + \frac{\partial f}{\partial y}\frac{dy}{d\alpha} + \frac{\partial f}{\partial \alpha} = 0$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

or $$\frac{\partial f}{\partial x}.\phi'(\alpha)+\frac{\partial f}{\partial y}\psi'(\alpha)=0 \qquad \left[\because \frac{\partial f}{\partial \alpha}=0, \text{ by (3)}\right]$$

or $$\frac{\psi'(\alpha)}{\phi'(\alpha)}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \qquad \ldots(5)$$

Now R.H.S. of (5) is the slope of the tangent at an ordinary point (x, y) of a curve 'α' of the family, and the L.H.S. is the slope of the tangent at the same point to the envelope (4). These slopes being equal, the curve 'α' and the envelope have the same tangent at the common point and so they touch.

Hence the result.

Note 1. If for any point on a curve $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are both zero, then the above argument fails, so that the envelope may not touch a curve at points which are the singular points on the curve.

Note 2. We know that a straight line and a conic cannot have a singular point. So the envelope of a family of straight lines or of conics touches each member of the family at all their common points without exception.

ILLUSTRATIVE EXAMPLES

Example 1. Find the envelope of the family

$$x^2(x-a)+(x+a)(y-m)^2=0$$

where a is a constant and m is a parameter.

Sol. The equation of family of curves is

$$x^2(x-a)+(x+a)(y-m)^2=0 \qquad \ldots(1)$$

Differentiating (1) partially w.r.t. m, we get,

$$0+2(x+a)(y-m)(-1)=0$$

$$\therefore \quad y-m=0 \quad \Rightarrow \quad m=y$$

Eliminating m from (1) and (2), we get,

$$x^2(x-a)+(x+a)(y-y)^2=0$$

or $$x^2(x-a)=0$$

which is the required envelope.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Example 2. Find the envelope of the family of lines

$$y = m\,x + \sqrt{a^2 m^2 + b^2}\ ,\ m \text{ being the parameter.}$$

Sol. The equation of the family of lines is

$$y = m\,x + \sqrt{a^2 m^2 + b^2}$$

or $\quad y - m\,x = \sqrt{a^2 m^2 + b^2}$

Squaring , $y^2 - 2\,m\,x\,y + m^2\,x^2 = a^2\,m^2 + b^2$

or $\quad (x^2 - a^2)\,m^2 - 2\,m\,x\,y + (y^2 - b^2) = 0$

which is a quadratic in parametric

Equating to zero the discriminant of (1), we get

$$(-2\,x\,y)^2 - 4\,(x^2 - a^2)\,(y^2 - b^2) = 0 \qquad [b^2 - 4\,a\,c = 0]$$

or $\quad x^2\,y^2 - (x^2 - a^2)\,(y^2 - b^2) = 0$

or $\quad x^2\,y^2 - x^2\,y^2 + b^2\,x^2 + a^2\,y^2 - a^2\,b^2 = 0$

or $\quad b^2\,x^2 + a^2\,y^2 = a^2\,b^2$

or $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

which is required envelope.

Example 3. Find the envelope of the family of circles

$$x^2 + y^2 - 2\,a\,x\cos\alpha - 2\,a\,y\sin\alpha = c^2$$

Sol. The equation of family of circles

$$x^2 + y^2 - 2\,y\cos\alpha - 2\,a\,y\sin\alpha = c^2$$

or $\quad 2\,a\,x\cos\alpha + 2\,a\,y\sin\alpha = x^2 + y^2 - c^2 \qquad \ldots(1)$

Differentiating (1) partially w.r.t. α , we get,

$$-2\,a\,x\sin\alpha + 2\,a\,y\cos\alpha = 0 \qquad \ldots(2)$$

Squaring and adding (1) and (2), we get,

$$4\,a^2\,x^2\,(\cos^2\alpha + \sin^2\alpha) + 4\,a^2\,y^2\,(\sin^2\alpha + \cos^2\alpha)$$

$$= (x^2 + y^2 - c^2)^2$$

or $\quad 4\,a^2\,x^2\,(1) + 4\,a^2\,y^2\,(1) = (x^2 + y^2 - c^2)^2$

or $\quad 4\,a\,(x^2 + y^2) = (x^2 + y^2 - c^2)^2$

which is required envelope.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Example 4. Find the envelope of the family of lines

$$x \cos^3 \theta + y \sin^3 \theta = a, \text{ where } \theta \text{ is parameter.}$$

Sol. The equation of the family of lines is

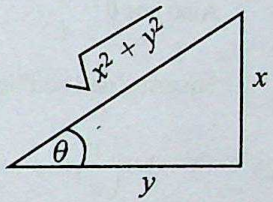
$$x \cos^3 \theta + y \sin^3 \theta = a \qquad \ldots(1)$$

Differentiating (1) partially w.r.t. θ, we get,

$$-3\, x \cos^2 \theta \sin \theta + 3\, y \sin^2 \theta \cos \theta = 0$$

or $\quad x \cos \theta - y \sin \theta = 0 \quad \Rightarrow \quad x \cos \theta = y \sin \theta$

$$\Rightarrow \quad \tan \theta = \frac{x}{y}$$



$$\therefore \quad \sin \theta = \frac{x}{\sqrt{x^2+y^2}}$$

and $\cos \theta = \dfrac{y}{\sqrt{x^2+y^2}}$

Substituting values of $\sin \theta$ and $\cos \theta$ in (1), we get,

$$x.\frac{y^3}{(x^2+y^2)^{\frac{3}{2}}} + y.\frac{x^3}{(x^2+y^2)^{\frac{3}{2}}} = a$$

or $\quad x\, y\, (x^2+y^2) = a\, (x^2+y^2)^{\frac{3}{2}}$

or $\quad x\, y = a (x^2+y^2)^{\frac{1}{2}}$

Squaring , $x^2 y^2 = a^2 (x^2+y^2)$

Which is required envelope.

Another Form : Find the curve for which the straight line

$x \cos^3 \theta + y \sin^3 \theta = a$ is always a tangent, θ being a parameter.

Example 5. Find the envelope of the ellipses

$$x = a \sin (\theta - \alpha), y = b \cos \theta$$

where α is the parameter.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Sol. The equations of the ellipses are

$$x = a \sin(\theta - \alpha) \qquad \ldots(1)$$

and $y = b \cos\theta$ $\qquad \ldots(2)$

From (1), $x = a(\sin\theta \cos\theta - \cos\theta \sin\alpha)$

$$\therefore \quad x = a \sin\theta \cos\alpha - a \sin\alpha . \frac{y}{b} \qquad [\because \text{ of } (2)]$$

$$\therefore \quad \sin\theta = \frac{x}{a}\sec\alpha + \frac{y}{b}\tan\alpha$$

Also $\cos\theta = \dfrac{y}{b}$

Squaring and adding above two equations,

$$1 = \left(\frac{x}{a}\sec\alpha + \frac{y}{b}\tan\alpha\right)^2 + \frac{y^2}{b^2} \qquad \ldots(3)$$

which is the family of ellipses.

Differentiating (3) w.r.t. α, we get,

$$0 = 2\left(\frac{x}{a}\sec\alpha + \frac{y}{b}\tan\alpha\right)\left(\frac{x}{a}\sec\alpha\tan\alpha + \frac{y}{b}\sec^2\alpha\right)$$

$$\therefore \quad \left(\frac{x}{a}\sec\alpha + \frac{y}{b}\tan\alpha\right)\left(\frac{x}{a}\tan\alpha + \frac{y}{b}\sec\alpha\right) = 0$$

$$\therefore \quad \frac{x}{a}\sec\alpha + \frac{y}{b}\tan\alpha = 0 \qquad \ldots(4)$$

$$\text{or} \quad \frac{x}{a}\tan\alpha + \frac{y}{b}\sec\alpha = 0 \qquad \ldots(5)$$

Eliminating α from (3) and (4) we get,

$$1 = \frac{y^2}{b^2} \quad \text{or } y = \pm b \qquad \ldots(6)$$

Now we eliminate α from (3) and (5)

From (5), $\dfrac{x \sin\alpha}{a\cos\alpha} + \dfrac{y}{b\cos\alpha} = 0$

$$\therefore \quad bx\sin\alpha + ay = 0 \quad \Rightarrow \sin\alpha = -\frac{ay}{bx} \qquad \ldots(7)$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

From (3), $1 = \frac{x^2}{a^2}\sec^2\alpha + \frac{2xy}{ab}\sec\alpha\tan\alpha + \frac{y^2}{b^2}(\tan^2\alpha + 1)$

$$\therefore \quad 1 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\sec^2\alpha + \frac{2xy\sin\alpha}{ab\cos^2\alpha}$$

$$\therefore \quad \cos^2\alpha = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2xy}{ab}\sin\alpha$$

$$\therefore \quad 1 - \sin^2\alpha = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2xy}{ab}\sin\alpha$$

$$\therefore \quad 1 - \frac{a^2y^2}{b^2x^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2y^2}{b^2} \qquad [\because \text{ of } (7)]$$

$$\therefore \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{a^2y^2}{b^2x^2} = 1 \qquad \ldots(8)$$

(6) and (8) constitute the envelope.

Example 6. From any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, perpendiculars are drawn to the axes, and the feet of these perpendiculars are joined. Show that the straight line thus formed always touches the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1.$$

Sol. Let P ($a\cos\theta$, $b\sin\theta$) be any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and M, N the feet of perpendiculars from P on the major axis (x-axis), minor axis (y-axis). Then M is ($a\cos\theta$, 0) and N is (0, $b\sin\theta$).

The equation of straight line MN is

$$\frac{x}{a\cos\theta} + \frac{y}{b\sin\theta} = 1 \qquad \ldots(1)$$

[$\because$ line makes intercepts $a\cos\theta$, $b\sin\theta$]

Differentiating (1) partially w.r.t. θ, we get,

$$\frac{x\sin\theta}{a\cos^2\theta} - \frac{y\cos\theta}{b\sin^2\theta} = 0$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

$$\therefore \quad \frac{x\sin\theta}{a\cos^2\theta} = \frac{y\cos\theta}{b\sin^2\theta}$$

$$\Rightarrow \quad \frac{\sin^3\theta}{\cos^3\theta} = \frac{ay}{bx} = \frac{\frac{y}{b}}{\frac{x}{a}}$$

$$\therefore \quad \frac{\cos\theta}{\left(\frac{x}{a}\right)^{\frac{1}{3}}} = \frac{\sin\theta}{\left(\frac{y}{b}\right)^{\frac{1}{3}}} = \pm\frac{1}{\sqrt{\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}}}$$

$$\therefore \quad \cos\theta = \pm\frac{\left(\frac{x}{a}\right)^{\frac{1}{3}}}{\sqrt{\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}}}, \sin\theta = \pm\frac{\left(\frac{y}{b}\right)^{\frac{1}{3}}}{\sqrt{\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}}}$$

Substituting these values of $\sin\theta$, $\cos\theta$ in (1), we get,

$$\pm\frac{x}{a}\frac{\sqrt{\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}}}{\left(\frac{x}{a}\right)^{\frac{1}{3}}} \pm\frac{y}{b}\frac{\sqrt{\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}}}{\left(\frac{y}{b}\right)^{\frac{1}{3}}} = 1$$

or
$$\pm\left[\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}} = 1$$

or
$$\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$$

which is required envelope.

Example 7. Find the envelope of the circles which pass through the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and their centres are upon its circumference.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Sol. The equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Its centre is C (0, 0) and any point on it is P ($a\cos\theta$, $b\sin\theta$)

$\therefore \quad CP^2 = (a\cos\theta - 0)^2 + (b\sin\theta - 0)^2$

$= a^2\cos^2\theta + b^2\sin^2\theta$

The equation of circles with centre as P ($a\cos\theta$, $b\sin\theta$) and radius CP is

$$(x - a\cos\theta)^2 + (y - b\sin\theta)^2 = a^2\cos^2\theta + b^2\sin^2\theta$$

or $\quad x^2 + y^2 - 2\,ax\cos\theta - 2\,by\sin\theta = 0$

or $\quad 2\,ax\cos\theta + 2\,by\sin\theta = x^2 + y^2 \qquad \ldots(1)$

Differentiating (1) partially w.r.t. θ, we get,

$$-2\,ax\sin\theta + 2\,by\cos\theta = 0 \qquad \ldots(2)$$

Squaring and adding (1) and (2), we get,

$$4\,a^2x^2 + (\cos^2\theta + \sin^2\theta) + 4\,b^2y^2(\sin^2\theta + \cos^2\theta) = (x^2 + y^2)$$

or $\quad 4\,(a^2x^2 + b^2y^2) = (x^2 + y^2)^2$

which is required envelope

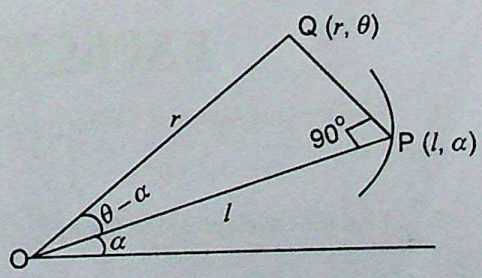
Example 8. Find the envelope of the straight lines drawn through extremities of and at right angles to the radii vectors of the following curve :

$$r = a\,(1 + \cos\theta)$$

Sol. Let P (l, α) be any point on the given curve

$$r = a\,(1 + \cos\theta)$$

$\therefore \quad l = a\,(1 + \cos\alpha) \qquad \ldots(1)$



Let Q (r, θ) be any point on the line drawn at right angles to the radius vector OP.

In right angled ΔOPQ,

$$\frac{l}{r} = \cos(\theta - \alpha) \quad \text{or} \quad l = r\cos(\theta - \alpha) \qquad \ldots(2)$$

which is the equation of line PQ.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Eliminating l from (1) and (2), we get,

$$r \cos(\theta - \alpha) = a(1 + \cos\alpha)$$

Taking logarithms,

$$\log r + \log\cos(\theta - \alpha) = \log a + \log(1 + \cos\alpha)$$

Differentiating partially w.r.t. α, we get,

$$0 + \frac{1}{\cos(\theta - \alpha)}\{-\sin(\theta - \alpha)\}(-1) = 0 + \frac{1}{1 + \cos\alpha}.(-\sin\alpha)$$

$$\therefore \quad \tan(\theta - \alpha) = -\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}}$$

$$\Rightarrow \quad \tan(\theta - \alpha) = -\tan\alpha$$

$$\Rightarrow \quad \tan(\theta - \alpha) = \tan\left(-\frac{\alpha}{2}\right)$$

$$\Rightarrow \quad \theta - \alpha = -\frac{\alpha}{2} \qquad \Rightarrow \quad 2\theta - 2\alpha = -\alpha$$

$$\Rightarrow \quad \alpha = 2\theta$$

Substituting in (3), we get,

$$r\cos(-\theta) = a(1 + \cos 2\theta)$$

or $\quad r\cos\theta = 2a\cos^2\theta$

or $\quad r = 2a\cos\theta$

which is required envelope.

EXERCISE 16 (*a*)

1. Find the envelope of the family of semi-cubical parabolas

$$y^2 - (x + \alpha)^3 = 0.$$

2. Find the envelope of the family

(*i*) $y = mx + \dfrac{a}{m}$, the parameter being m

(*ii*) $y = mx + am^p$, the parameter being m

(*iii*) $(x - \alpha)^2 + y^2 = 4\alpha$, the parameter being α

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

(*iv*) $a y^2 = (x - \alpha)^3$, the parameter being α

(*v*) $(y + \alpha)^2 = (x + 1)$ the parameter being α

(*vi*) $(x - \alpha)^2 + y^2 = \alpha^2$, the parameter being α

3. Find the envelope of the family of lines

(*i*) $y = m x + \dfrac{a}{m}$

(*ii*) $y = m x + a\sqrt{1 + m^2}$

4. Find the envelope of the family of curves

$$\frac{x^2}{\lambda^2} + \frac{y^2}{a^2 - \lambda^2} = 1, \lambda \text{ being parameter.}$$

5. Find the envelope of the family of curves

$$y = x \tan\theta - \frac{g x^2}{2 u^2 \cos^2\theta} \text{ parameter } \theta.$$

6. Find the envelope of the family of curves

$$\frac{a^2}{x}\cos\theta - \frac{b^2}{y}\sin\theta = c$$

for different values of θ.

7. Find the envelope of the family of lines

(*i*) $x \cos\alpha + y \sin\alpha = p$, parameter α

(*ii*) $x \cos\theta + y \sin\theta = a(1 + \cos\theta)$, parameter θ

(*iii*) $x \operatorname{cosec}\theta - y \cot\theta = 0$, parameter θ.

8. Find the envelope of the family of lines

(*i*) $x \cos\theta + y \sin\theta = l \sin\theta \cos\theta$, θ being the parameter.

(*ii*) $\dfrac{a x}{\cos\theta} - \dfrac{b y}{\sin\theta} = a^2 - b^2 + \theta$ being the parameter.

(*iii*) $x \cos^n\theta + y \sin^n\theta = a$, θ being the parameter,

9. (*i*) Find the envelope of the family of lines

$$a x \cos\theta + b y \cot\theta = a^2 + b^2, \theta \text{ being the parameter.}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

(*ii*) Find the envelope of the family of curves

$$\frac{x^3}{a^3 \cos\theta} + \frac{y^3}{b^3 \sin\theta} = 1, \theta \text{ being the parameter.}$$

10. Show that the envelope of the family of curves

$$A\lambda^3 + 3B\lambda^2 + 3C\lambda + D = 0$$

where λ is the parameter and A, B, C, D are functions of x and y, is

$$(BC - AD)^2 = 4(BD - C)^2 (AC - B^2)$$

11. Find the envelope of the family of straight lines

$$a\cos\alpha + y\sin\alpha = f(\alpha)$$

where α is the parameter.

Also show that the radius of curvature of the envelope is

$$f(\alpha) + f''(\alpha).$$

12. Show that the envelope of the polar of the points on the ellipse $\frac{x^2}{h^2} + \frac{y^2}{k^2} = 1$ w.r.t the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the ellipse $\frac{h^2x^2}{a^4} + \frac{k^2y^2}{b^4} = 1$.

13. A straight line of given length slides with its extremities on two fixed straight lines at right angles. Find the envelope of the circle drawn on the sliding line as diameter.

14. Show that the envelope of the straight lines of given length l, which slide with extremities on two fixed straight lines at right angles is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = l^{\frac{2}{3}}$.

15. Show that the envelope of a circle whose centre lies on parabola $y^2 = 4ax$ and which passes through its vertex, is

$$2ay^2 + x(x^2 + y^2) = 0.$$

16. Find the envelope of the circles, whose diameter is the radius vector of the parabola $y^2 = 4ax$.

17. Show that the envelope of the family of circles whose diameters are the double ordinates of the parabola $y^2 = 4ax$ is the parabola $y^2 = 4a(x + a)$.

18. Find the envelope of the circles whose diameters are double ordinates of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

 Gandhi Memorial College Of Education Bantalab Jammu

19. Find the envelope of the circles drawn upon the central radii vectors of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as diameter.

20. Show that the envelope of the straight lines joining the extremities of a pair of conjugate diameter of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$.

21. Find the envelope of the family of parabolas of given latus rectum and parallel axes, when the locus of their foci is a given straight line $y=px+q$.

22. Find the envelope of the straight lines drawn through extremities of and at right angles to the radii vectors of the following curves $r^n = a^n \cos n\theta$.

ANSWERS

1. $y=0$

2. (*i*) $y^2=4ax$ (*ii*) $(p-1)^{p-1}x^p+p^p a y^{p-1}=0$

(*iii*) $y^2=4x+4$ (*iv*) $y=0$

(*v*) $x^2(x+1)=0$ (*vi*) no envelope.

3. (*i*) $y^2=4ax$ (*ii*) $x^2+y^2=a^2$

4. $(x-y-a)(x+y-a)(x-y+a)(x+y+a)=0$

5. $y+\frac{gx^2}{2u^2}=\frac{u^2}{2g}$ **6.** $\frac{a^4}{x^2}+\frac{b^4}{y^2}=c^2$

7. (*i*) $x^2+y^2=p^2$ (*ii*) $(x-3)^2+y^2=a^2$ (*iii*) $x^2-y^2=c^2$

8. (*i*) $\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^3=l^2$ (*ii*) $\left[(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}\right]^3=(a^2-b^2)^2$

(*iii*) $(xy)^{\frac{2}{2-n}}\left(x^{\frac{2}{n-2}}+y^{\frac{2}{n-2}}\right)=a^{\frac{2}{2-n}}$

9. (*i*) $\left[(ax)^{\frac{2}{3}}-(by)^{\frac{2}{3}}\right]^3=(a^2+b^2)^2$ (*ii*) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

11. $x = f(\alpha)\cos\alpha - f'(\alpha)\sin\alpha$, $y = f(\alpha)\sin\alpha + f'(\alpha)\cos\alpha$

13. $x^2 + y^2 = a^2$

16. $a y^2 + x(x^2 + y^2) = 0$

18. $\dfrac{x^2}{a^2 + b^2} + \dfrac{y^2}{b^2} = 1$

19. $a^2 x^2 + b^2 y^2 = (x^2 + y^2)^2$

21. $p^2 x - p y + p^2 a + p q + a = 0$

22. $r^{\frac{n}{1-n}} = a^{\frac{n}{1-n}} \cos\dfrac{n\theta}{1-n}$

Art-7. Two Parameters connected by a Relation

Explain the method to find the envelope of a family of curves $f(x, y, \alpha, \beta) = 0$ where α, β are connected by the relation $\phi(\alpha, \beta) = 0$.

Proof. The equation of family of curves is

$$f(x, y, \alpha, \beta) = 0 \qquad \ldots(1)$$

where $$\phi(\alpha, \beta) = 0 \qquad \ldots(2)$$

Method I. If convenient, eliminate α or β between (1) and (2), getting the equation of the family in terms of one parameter. Then proceed as usual.

Method II. Regard β as a function of α. Differentiate (1) and (2) w.r.t. α, regarding x and y as constants so that

$$\frac{\partial f}{\partial \alpha} + \frac{\partial f}{\partial \beta} \cdot \frac{d\beta}{d\alpha} = 0 \qquad \ldots(3)$$

and $$\frac{\partial \phi}{\partial \alpha} + \frac{\partial \phi}{\partial \beta} \cdot \frac{d\beta}{d\alpha} = 0 \qquad \ldots(4)$$

Eliminating $\alpha, \beta, \dfrac{d\beta}{d\alpha}$ from (1), (2), (3) and (4), we get the required envelope.

ILLUSTRATIVE EXAMPLES

Example 1. Find the envelope of the family of ellipses $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where the two parameters a, b are connected by the relation $a + b = c$; c, being a constant.

Sol. The equation of the family of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \ldots(1)$$

where $$a + b = c$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

From (2), $b = c - a$

Putting $b = c - a$ in (1), we get,

$$\frac{x^2}{b^2} + \frac{y^2}{(c-a)^2} = 1 \qquad \ldots(3)$$

which is the equation of the family involving one parameter a.

Differentiating (3) partially w.r.t. a, we get,

$$-\frac{2x^2}{a^3} + \frac{2y^2}{(c-a)^2} = 0, \quad \text{or} \quad \frac{c-a}{a} = \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}$$

$$\therefore \quad \frac{(c-a)+a}{a} = \frac{y^{\frac{2}{3}} + x^{\frac{2}{3}}}{x^{\frac{2}{3}}} \quad \Rightarrow \quad \frac{c}{a} = \frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{x^{\frac{2}{3}}}$$

$$\therefore \quad a = \frac{c\,x^{\frac{2}{3}}}{x^{\frac{2}{3}} + y^{\frac{2}{3}}}$$

$$c - a = c - \frac{c\,x^{\frac{2}{3}}}{x^{\frac{2}{3}} + y^{\frac{2}{3}}} = \frac{c\,y^{\frac{2}{3}}}{x^{\frac{2}{3}} + y^{\frac{2}{3}}}$$

Substituting the values of a, $c - a$ in (3), we get,

$$\frac{x^2\left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right)^2}{c^2\,x^{\frac{4}{3}}} + \frac{y^2\left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right)^2}{c^2\,y^{\frac{4}{3}}} = 1$$

or $$x^{\frac{2}{3}}\left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right)^2 + y^{\frac{2}{3}}\left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right)^2 = c^2$$

or $$\left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right)^2\left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right) = c^2 \quad \text{or} \quad \left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right)^3 = c^2$$

or $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$, which is required envelope.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College of Education Bantala Kolkata

Another form. Prove that the envelope of ellipses having the axes of co-ordinates as principal axes and sum of their semi-axes constant and equal to c, is the astroid

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}.$$

Example 2. Find the envelope of the family of lines $\frac{x}{a} + \frac{y}{b} = 1$ where a, b are connected by the relation

(*i*) $a + b = c$

(*ii*) $a^2 + b^2 = c^2$

(*iii*) $ab = c^2$

Sol. (*i*) The equation of the family of lines is

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \ldots(1)$$

where $a + b = c$ $\qquad \ldots(2)$

Regard b as a function of a as is clear from (2).

Differentiating (1) and (2) partially w.r.t. a,

$$-\frac{x}{a^2} - \frac{y}{b^2}\frac{db}{da} = 0 \qquad \ldots(3)$$

and $$1 + \frac{db}{da} = 0 \qquad \ldots(4)$$

From (4), $$\frac{db}{da} = -1$$

Substituting the value of $\frac{db}{da}$ in (3), we get

$$-\frac{x}{a^2} + \frac{y}{b^2} = 0, \quad \text{or} \quad \frac{x}{a^2} = \frac{y}{b^2}$$

or $$\frac{\frac{x}{a}}{a} = \frac{\frac{y}{b}}{b} = \frac{\frac{x}{a} + \frac{y}{b}}{a+b} = \frac{1}{c} \qquad [\because \text{ of (1), (2)}]$$

$$\therefore \quad x = \frac{a^2}{c}, y = \frac{b^2}{c}$$

$$\therefore \quad a^2 = cx, b^2 = cy \quad \Rightarrow \quad a = c^{\frac{1}{2}} x^{\frac{1}{2}}, \; b = c^{\frac{1}{2}} y^{\frac{1}{2}}$$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

 Gandhi Memorial College Of Education Bantalab Jammu

Substituting values of a, b in (2), we get,

$$c^{\frac{1}{2}} x^{\frac{1}{2}} + c^{\frac{1}{2}} y^{\frac{1}{2}} = c$$

or $$x^{\frac{1}{2}} + y^{\frac{1}{2}} = c^{\frac{1}{2}}$$

which is required envelope.

(*ii*) The equation of the family of lines is

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \ldots(1)$$

where $$a^2 + b^2 = c^2 \qquad \ldots(2)$$

Regard b as a function of a as is clear from (2).

Differentiating (1) and (2) partially w.r.t. a, we get,

$$-\frac{x}{a^2} - \frac{y}{b^2}\frac{db}{da} = 0 \qquad \ldots(3)$$

and $$2a + 2b\frac{db}{da} = 0 \qquad \ldots(4)$$

from (4), $$\frac{db}{da} = -\frac{a}{b}$$

Substituting value of $\frac{db}{da}$ in (3), we get,

$$-\frac{x}{a^2} - \frac{y}{b^2}\left(-\frac{a}{b}\right) = 0, \quad \text{or} \quad \frac{x}{a^3} = \frac{y}{b^3}$$

$$\therefore \quad \frac{\frac{x}{a}}{a^2} = \frac{\frac{y}{b}}{b^2} = \frac{\frac{x}{a} + \frac{y}{b}}{a^2 + b^2} = \frac{1}{c^2} \qquad [\because \text{ of } (1), (2)]$$

$$\therefore \quad \frac{x}{a^3} = \frac{1}{c^2}, \frac{y}{b^3} = \frac{1}{c^2} \quad \Rightarrow \quad a^3 = c^2 x, \; b^3 = c^2 y$$

$$\therefore \quad a = c^{\frac{2}{3}} x^{\frac{1}{3}}, \; b = c^{\frac{2}{3}} y^{\frac{1}{3}}$$

Substituting values of a, b in (2), we get,

$$c^{\frac{4}{3}} x^{\frac{2}{3}} + c^{\frac{4}{3}} y^{\frac{2}{3}} = c^2$$

or $$x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$$

which is required envelope.

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

(*iii*) The equation of the family of lines is

$$\frac{x}{a}+\frac{y}{b}=1 \qquad \ldots(1)$$

where $\quad a\,b=c^2 \qquad \ldots(2)$

Regard b as a function of a as is clear from (2).

Differentiating (1) and (2) partially w.r.t. a, we get

$$-\frac{x}{a^2}-\frac{y}{b^2}\frac{db}{da}=0 \qquad \ldots(3)$$

and $\quad b+a\dfrac{db}{da}=0 \qquad \ldots(4)$

From (4), $\quad \dfrac{db}{da}=-\dfrac{b}{a}$

Putting value of $\dfrac{db}{da}$ in (3), we get,

$$-\frac{x}{a^2}+\frac{y}{b^2}.\frac{b}{a}=0 \quad \Rightarrow \quad \frac{x}{a}=\frac{y}{b}$$

$$\therefore \quad \frac{\frac{x}{a}}{1}=\frac{\frac{y}{b}}{1}=\frac{\frac{x}{a}+\frac{y}{b}}{1+1}=\frac{1}{2}$$

$$\therefore \quad \frac{x}{a}=\frac{1}{2},\ \frac{y}{b}=\frac{1}{2} \quad \Rightarrow \quad a=2\,x,\ b=2\,y$$

Substituting values of a, b in (2), we get,

$$(2\,x)\,(2\,y)=c^2 \quad \text{or} \quad 4\,x\,y=c^2$$

which is required envelope.

Example 3. Find the envelope of family of ellipses $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ such that $a^2+b^2=c$.

Sol. The equation of family of ellipses is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \ldots(1)$$

where $a^2+b^2=c \qquad \ldots(2)$

Regard b as a function of a as is clear from (2).

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

Differentiating (1) and (2) partially w.r.t. a, we get,

$$-\frac{2x^2}{a^3}-\frac{2y^2}{b^3}=0 \qquad \ldots(3)$$

and $$2a+2b\frac{db}{da}=0 \qquad \ldots(4)$$

From (4), $\frac{db}{da}=-\frac{a}{b}$

$\therefore$ from (3), $-\frac{2x^2}{a^3}+\frac{2y^2}{b^3}\cdot\frac{a}{b}=0$, or $\frac{x^2}{a^4}=\frac{y^2}{b^4}$

$$\therefore \quad \frac{\frac{x^2}{a^2}}{a^2}=\frac{\frac{y^2}{b^2}}{b^2}=\frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}}{a^2+b^2}=\frac{1}{c} \qquad [\because \text{ of (1), (2)}]$$

$$\therefore \quad \frac{x^2}{a^4}=\frac{1}{c},\ \frac{y^2}{b^4}=\frac{1}{c} \quad \Rightarrow \quad a^4=cx^2,\ b^4=cy^2$$

$$\therefore \quad a^2=\sqrt{c}\,x, b^2=\sqrt{c}\,y$$

Substituting these values of a, b in (2), we get,

$$\sqrt{c}\,(x+y)=c \quad \text{or} \quad x+y=\sqrt{c}$$

which is required envelopes.

Another Form. Find the envelope of the ellipses having the axes of co-ordinates as principal axes and the sum of squares of their semi-axes constant.

EXERCISE 16 (*b*)

1. Find the envelope of the curves $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$, where the parameters a and b are connected by the relation $a^p+b^p=c^p$.

2. Find the envelope of the family of lines $\frac{x}{a}+\frac{y}{b}=1$ where a, b are connected by the relation

   (*i*) $a^3+b^3=c^3$ (*ii*) $a^n+b^n=c^n$ (*iii*) $a^m b^n=c^{m+n}$

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

3. Find the envelope of the ellipses having the axes of co-ordinates as principal axes and the semi-axes a, b connected by the relation $a\,b = c^2$.

4. Find the envelope of a system of concentric and co-axial ellipses of constant area.

5. Show that the envelope of the family of parabolas $\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1$ under the condition $a\,b = c^2$ is a hyperbola having its asymptotes coinciding with the axes.

ANSWERS

1. $x^{\frac{mp}{m+p}} + y^{\frac{mp}{m+p}} = c^{\frac{mp}{m+p}}$

2. (*i*) $x^{\frac{3}{4}} + y^{\frac{3}{4}} = c^{\frac{3}{4}}$ (*ii*) $x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} = c^{\frac{n}{n+1}}$

(*iii*) $(m+n)^{m+n} x^m y^n = m^m n^n c^{m+n}$

3. $2\,x\,y = c^2$

4. $2\,x\,y = c^2$.

J.M. College of Education
Raipur, Bantalab
Jammu.
Acc No....3403(D)
Dated...25/11/202

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

CC-0 Agamnigam Digital Preservation Foundation,Chandigarh.Funding by IKS-MoE-2025-Grant

Gandhi Memorial College Of Education Bantalab Jammu

ISBN 93-5181-193-X
9 789351 811930

Agamnigam Digital Preservation Foundation,Chandigarh,Funding by IKS-MoE-2025-Grant